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Quantization, Deformations, and Symmetries

Volume II

Edited by

Giuseppe Dito
Daniel Sternheimer

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Quantization, Deformations, and Symmetries
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Giuseppe Dito
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Dijon, France

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Contributed papers

Singletons on AdS_n

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Abstract. We define the singletons for the invariance group $\overline{S}_n = \overline{SO}_0(2, n-1)$ of the AdS_n space-time. We write down some of their important properties and characterizations. It is found that the tensor product of singletons of spin 0 or 1/2 decomposes into representations that are a kind of massless representations of $\overline{S}_n$. Other kinds of massless representations, related to singletons, are also studied and a comparison is made. Various Gupta-Bleuler triplets are constructed for singletons and for massless representations.

Keywords: representations of Lie groups and Lie algebras; Minkowski and anti-de Sitter spaces; anti-de Sitter, conformal, and Poincaré groups; singletons; masslessness; Gupta-Bleuler triplets

Mathematics Subject Classifications (2000): 22E43, 22E46, 22E70, 81R05, 81T20, 81V25, 83E99

Dedicated to the memory of Moshé Flato

1. Introduction

Recently, mainly since the Maldacena conjecture which relates anti-de Sitter theories 'in the bulk' with conformal theories 'on the boundary at infinity', singletons became a widely popular subject in physics on space-times of arbitrary dimensions (see, e.g. [27, 11, 30, 17, 19, 1]). It is therefore important to know more about these representations and about massless representations of conformal groups. In this paper we describe some properties of singletons and characterize them after having given a rigorous definition. Since the singletons are related to masslessness it is important to know the nature of that relation and to compare with what happens in the classical 4-dimensional case.

The n-dimensional anti-de Sitter space-time (AdS_n) with (scalar) curvature $-\rho < 0$ is defined as the manifold $H_n^\rho = \{(y^a)_{-1 \le a \le n-1} \in \mathbb{R}^{n+1} / \sum y^a y_a = 1/\rho\}$. Here $\sum y^a y_a \overset{\mathrm{def}}{=} \sum y^a y^b \eta_{ab}$ where $\eta = (\eta_{ab})$ is the matrix $\left(\begin{array}{c|c} \mathbf{1}_2 & \\ \hline & -\mathbf{1}_n \end{array}\right)$. We assume throughout that $n \ge 3$. The invariance group of AdS_n is the anti-de Sitter group $S_n = SO_0(2, n-1)$ and one has $H_n^\rho \simeq S_n / L_n$, where $L_n = SO_0(1, n-1)$ is the Lorentz group of both AdS_n and Minkowski space-time $M_n = \mathbb{R}^{1,n-1}$.

Now the "time axis" of H_n^ρ is bounded: It is the S^1 (circle) part in $H_n^\rho \simeq S^1 \times \mathbb{R}^{n-1}$. But if one considers the universal covering $\overline{H}_n^\rho \simeq \mathbb{R} \times \mathbb{R}^{n-1}$ of H_n^ρ, then the time axis is no longer compact (of course there is no problem if one needs a physical theory with a cyclic time).

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Consequently, the invariance group is an infinite covering $\overline{S}_n$ of S_n. The best choice for $\overline{S}_n$ would be a covering which contains the quantum mechanical Lorentz group, i.e., the spinor covering group $\overline{L}_n$, such that $\overline{H}_n^\rho \simeq \overline{S}_n/\overline{L}_n$. Having in mind that an invariance group should be contained in the conformal group, such a choice is given by the universal covering of S_n if $n \geq 4$ and, for $n = 3$, by the infinite covering of $S_3 = SO_0(2,2)$, which is contained in, and has the same center as, the conformal group $\overline{G}_3 = \overline{SO}_0(2,3)$. Thus the fundamental group of $\overline{S}_n = \overline{SO}_0(2,n)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$ for all $n \geq 3$. Another advantage in using this type of infinite coverings is that we can consider deformations of representations, a useful tool for constructing Gupta-Bleuler (GB) triplets.

Let $\overline{G}_n = \overline{SO}_0(2,n)$ the common conformal group [1] of $\overline{H}_n^\rho$ and M_n. Then $\overline{G}_n$ contains the Poincaré group $\overline{P}_n = \overline{SO}_0(1,n-1) \ltimes T_n$ ($T_n \simeq \mathbb{R}^n$), the anti-de Sitter group $\overline{S}_n = \overline{SO}_0(2,n-1)$, and the de Sitter group $\overline{L}_{n+1} = \overline{SO}_0(1,n)$. It is well known that the last two are deformations of the former, so that $\overline{S}_n$ and $\overline{L}_{n+1}$ can be contracted to $\overline{P}_n$. It follows that there is a contraction $\overline{H}_n^\rho \xrightarrow{\rho \to 0} M_n$ which implies that a physical theory on AdS$_n$ cannot be independent of the corresponding one on M_n: It must be compatible, at least for $n = 4$, with physics in M_n. As a consequence, a massless particle on AdS$_n$ should correspond by contraction to a massless particle on M_n. This naturally leads to a first (weak) definition of masslessness (see Section 3).

Unfortunately this definition does not fix uniquely the notion of masslessness on AdS$_n$, even for $n = 4$: Additional conditions have to be introduced in order to make it unique. This was done by Flato and Frønsdal for $n = 4$ in the 80's (see [14]). Another way to avoid ambiguity is to consider less weak definitions of masslessness, introduced in Section 3, such as *conformal masslessness* or *composite masslessness*, both related to singletons and to gauge properties. The latter, as shown by Flato and Frønsdal in [13] (see also [23]), is expressed by the well-known property "*singleton* $\otimes$ *singleton* $= \oplus$ *massless representations*", while the former is (for $n \geq 3$) the property of unique extension from representations of the anti-de Sitter group to the corresponding conformal group. For $n = 4$, both notions coincide.

The paper is organized as follows. In Section 2, we define and characterize finite-dimensional (nonunitary) and infinite-dimensional (unitary) singleton representations of $\overline{SO}_0(2,n-1)$, along with a classification. We also obtain a generalization of the theorem of Flato and Frønsdal mentioned above. We define in Section 3 some notions of masslessness related to singletons, give their properties and discuss their differences. In section 4, we construct examples of Gupta-Bleuler triplets for the singletons and for almost all massless representations. We conclude the paper by a comparison between the 4-dimensional and higher-dimensional cases.

---

[1] In fact, $\overline{G}_n$ is a covering of the actual conformal group.

2. Singletons of $\overline{SO}_0(2,n-1)$

2.1. The fundamental property of singletons

Let $\mathcal{G}_n = \mathrm{Lie}(\overline{G}_n)$, $\mathcal{S}_n = \mathrm{Lie}(\overline{S}_n)$, $\mathcal{L}_n = \mathrm{Lie}(\overline{L}_n)$, and $\mathcal{P}_n = \mathrm{Lie}(\overline{P}_n)$. Let $(e_a)_{-1\leq a\leq n}$ be the canonical basis of $\mathbb{R}^{2,n}$ which is endowed with the metric η. Then a set of generators of $\mathcal{G}_n$ is given by $\{M_{ab}\}_{-1\leq a,b\leq n}$, defined by:

$$M_{ab} = -M_{ba} \text{ and } M_{ab}y = y_b e_a - y_a e_b \quad \forall\, y = \sum_{a=-1}^{a=n} y^a e_a \in \mathbb{R}^{2,n}.$$

The following commutation relations are satisfied:

$$[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} + \eta_{bd}M_{ca} - \eta_{ac}M_{bd} - \eta_{ad}M_{cb}. \tag{2.1}$$

$\mathcal{S}_n$, $\mathcal{L}_n$, and $\mathcal{P}_n$ are naturally embedded in $\mathcal{G}_n$. To obtain the generators of the first two Lie algebras one simply restricts the range of indices: $\{M_{\alpha\beta}\}_{-1\leq\alpha,\beta\leq n-1}$ and $\{M_{\mu\nu}\}_{0\leq\mu,\nu\leq n-1}$, respectively. $\mathcal{P}_n$ is the semi-direct sum of $\mathcal{L}_n$ and the Abelian Lie algebra $\mathcal{T}_n = \mathrm{Lie}(T_n)$, for which a set of generators can be given by $\{E_\mu = M_{-1,\mu} + M_{\mu,n}\}_{0\leq\mu\leq n-1}$. They satisfy the commutation relations:

$$[E_\mu, E_\nu] = 0 \quad \text{and} \quad [M_{\mu,\nu}, E_\rho] = \eta_{\nu\rho}E_\mu - \eta_{\mu\rho}E_\nu. \tag{2.2}$$

The notation used in physics is related to the present one by: $P_\mu = -\sqrt{-1}E_\mu$, $L_{\mu\nu} = \sqrt{-1}M_{\mu\nu}$, and so on.

Let D be an irreducible representation (IR) of the AdS$_n$ group $\overline{S}_n = \overline{SO}_0(2,n-1)$ on a Banach space $\mathcal{H}$, not necessarily unitary. Let $K_n \simeq SO(2)\times SO(n-1)$ be the maximal compact subgroup of S_n and $\overline{K}_n \simeq \mathbb{R}\times \mathrm{Spin}(n-1)$ be the corresponding maximal essentially compact subgroup of $\overline{S}_n$. The common reductive Lie algebra $\mathfrak{k}_n$ is generated by $M_{-1,0}$ and $\{M_{ij}\}_{1\leq i,j\leq n-1}$, the latter generating the semi-simple part of $\mathfrak{k}_n$. The restriction $D|_{\overline{K}_n}$ is completely reducible, i.e., under the action of $D|_{\overline{K}_n}$ one has the direct sum decomposition:

$$\mathcal{H}^\infty = \oplus_\mu M(\mu)\otimes K(\mu), \tag{2.3}$$

where each μ is a highest weight (HW) relative to a given order of the roots of $\mathfrak{k}_n$, $K(\mu)$ is an irreducible $\mathfrak{k}_n$-module with weight μ and $M(\mu)$ is a trivial $\mathfrak{k}_n$-module the dimension of which is the multiplicity $m(\mu)$ of μ. $\mathcal{H}^\infty$ is the subspace of differentiable vectors. It is known that $\mathcal{H}^\infty$ is dense in $\mathcal{H}$. If π_μ is an IR of $\overline{K}_n$ with weight μ then the relation (2.3) may be written:

$$D|_{\overline{K}_n} = \oplus_\mu m(\mu)\pi_\mu. \tag{2.4}$$

Let us write $(\mu_1, \vec{\mu})$ for $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ where r is the rank of $\mathcal{S}_n^{\mathbb{C}}$ (r is the entire part of $\frac{n+1}{2}$). Let us call μ_1 the *energy part* (we choose $-\mu_1$ to be the *energy*) and $\vec{\mu}$ the *spinor part*. Then the diagram of such $\mathfrak{k}_n$'s HW μ has more than one dimension in general (recall that $n \geq 3$). Moreover the multiplicity of each weight can be different from 0 or 1. But it may also happen that this diagram is just one dimensional. It is indeed the case for Dirac singletons and the so-called ladder representations of the conformal group [3, 8, 26, 29] or the representations of $\overline{G}_n$ called C_n-massless in [25].

When that diagram is included in a line then each weight may be obtained from a fixed one by adding an integer multiple of a fixed root. Thus the convex envelope of the diagram is one dimensional if the representation is not trivial. So let us write down a definition of the singletons. This definition generalizes the one given by Dirac [8] in the 4-dimensional case to the representations $D(s+1,s)$, s being $1/2$ or 0, respectively called later on *Di* and *Rac* by Flato and Frønsdal (see for example [3]). More generally, the notation $D(E,\vec{\lambda})$ corresponds to the irreducible representation (up to equivalence) carried by the irreducible quotient $L(-E,\vec{\lambda})$ of the Verma module $M(-E,\vec{\lambda})$.

Definition 2.1 *An IR D of $\overline{S}_n$ is a* singleton representation, *or more simply a* singleton, *if D is not trivial and there exists a weight λ and a root α of $\mathfrak{k}_n$ such that*

$$D|_{\overline{K}_n} = \oplus_{l \in \mathbb{Z}} \pi_{\lambda + l\alpha}, \tag{2.5}$$

where π_μ is 0, if μ is not a weight of $D|_{\overline{K}_n}$.

An example is given by Dirac singletons ($n = 4$) $D(s+1,s)$ and their contragredients $\overline{D}(s+1,s)$, identified with $D(-(s+1),s)$. Here one has:

$$D(\pm(s+1/2),s)|_{\overline{K}_4} = \oplus_{l \in \mathbb{N}} \pi_{(\pm(s+1/2+l),s+l)}$$

and $K_4 \simeq SO(2) \times SO(3)$.

In Theorem 2.2, we shall give the fundamental mathematical property of singletons. It is a strong property of the enveloping algebra $\mathscr{U}$ of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(N)^{\mathbb{C}}$. We introduce N in order to treat both conformal and anti-de Sitter cases, hence $\mathfrak{g}^{\mathbb{C}}$ stands for $\mathscr{S}_{N-1}{}^{\mathbb{C}}$ or $\mathscr{G}_{N-2}{}^{\mathbb{C}}$. Before stating the theorem let us introduce some useful notation. Let $M_N(\mathscr{U})$ be the vector space of $N \times N$ matrices whose elements are in $\mathscr{U}$. $M_N(\mathscr{U})$ is also endowed with a natural $\mathscr{U}$-module structure. Let $\delta = (\eta_{ab}\mathrm{Id})_{1 \leq a,b \leq N}$ and $M = (M_{ab})_{1 \leq a,b \leq N}$ be two such matrices, Id and the M_{ab}'s being respectively the identity of $\mathscr{U}$ and the generators of the Lie algebra $\mathfrak{so}(N)^{\mathbb{C}}$. The commutation relations considered are those given by (2.1). Define $M^0 \overset{\text{def}}{=} \delta$ and M^k for nonzero $k \in \mathbb{N}$ by $(M^k)_{ab} = \sum_{c=1}^{N} (M^{k-1})_{ac} M^c{}_b$ where $M^c{}_b \overset{\text{def}}{=} \sum_{d=1}^{N} \eta^{cd} M_{db}$. If D is a representation of $\mathscr{U}$ and $A = (A_{ab})$ an element of $M_N(\mathscr{U})$, we write $D(A)$ for the matrix having entries $D(A_{ab})$. Finally let $\mathbf{C}_2 = \frac{1}{2}\mathrm{Tr}(M^2)$ be the Casimir operator.

Theorem 2.2 *A singleton representation D of $G = \overline{SO}(2,N-2)$ is a highest or lowest weight representation and it satisfies* [2]

$$D\Big(M^2 - \frac{N-2}{2}M - \frac{2}{N}C_2\delta\Big) = 0. \tag{2.6}$$

Moreover D is unitarizable if and only if it is infinite dimensional.

Conversely a representation integrable on the maximal compact subalgebra $\mathfrak{k}$ of $\mathfrak{g} = Lie(G)$ which sends the two-sided ideal spanned by the family $(F_{ab})_{a,b}$ to 0 has, as irreducible parts, all the singleton representations.

Proof. Case 1: D is finite dimensional. Suppose that D is a finite-dimensional representation of $\overline{SO}(N)$ of weight $\lambda = (\lambda_1,\dots,\lambda_r)$, r being the rank. Let us denote by D'_μ and D''_ν the finite-dimensional representations of respective weights μ and $\nu = (\nu_2,\dots,\nu_r)$ (up to equivalence) of $\overline{SO}(N-1)$ and $\overline{SO}(N-2)$ respectively. Let us write $\mu = (\mu_1,\dots,\mu_r)$ if N is odd and $\mu = (\mu_2,\dots,\mu_r)$ if N is even. Then it is known that

$$D|_{\overline{SO}(N-1)} = \begin{cases} \oplus_{\lambda_1\geq\mu_2\geq\lambda_2\geq\cdots\geq\lambda_{r-1}\geq\mu_r\geq|\lambda_r|} D'_\mu & \text{if } N \text{ is even,} \\ \oplus_{\lambda_1\geq\mu_1\geq\lambda_2\geq\cdots\geq\lambda_r\geq\mu_r\geq-\lambda_r} D'_\mu & \text{if } N \text{ is odd.} \end{cases} \tag{2.7}$$

The same decomposition holds for $D'_\mu|_{\overline{SO}(N-2)}$. Thus $(D|_{\overline{SO}(N-1)})|_{\overline{SO}(N-2)}$ decomposes into a sum of irreducible representations D''_ν such that $\lambda_1 \geq \nu_2 \geq \lambda_2$ and so on.

Now thanks to the preceding relations one sees that for each index i such that $1 \leq i \leq r-1$, $\lambda_i > |\lambda_{i+1}|$ implies that the representation D'_{μ_i} occurs at least twice in the reduction $D|_{\overline{SO}(N-1)}$, thus D''_{ν_i} also occurs at least twice in $(D|_{\overline{SO}(N-1)})|_{\overline{SO}(N-2)}$. But if D is a singleton then what precedes must contain at most one component of D''_{ν_i} since after restriction to $\overline{K}$ (recall that $K \simeq SO(2)\times SO(N-2)$) one gets a sum of irreducible representations of the form $\chi(\nu_1)\otimes D''_\nu$ such that the multiplicities of ν_1 and ν are both 1. Thus one has necessarily $\lambda_1 = \cdots = \lambda_{r-1} = |\lambda_r|$ and this is equivalent, as it is proved in [4], to the relation (2.6).

Case 2: D is infinite dimensional. The Cartan decomposition of $\mathfrak{g} = \mathfrak{so}(2,N-2)$ writes $\mathfrak{k}+\mathfrak{p}$ and the triangular one is given by $\mathfrak{g}^{\mathbb{C}} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+$. A common Cartan subalgebra $\mathfrak{h}$ to $\mathfrak{k}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$ is generated by $H_1 = -\sqrt{-1}M_{-1,0}$ and, for j running from 2 to the rank r, $H_j = \sqrt{-1}M_{2j-3,2j-2}$. We write $(\varepsilon_j)_{1\leq j\leq r}$ for the dual basis of $(H_j)_{1\leq j\leq r}$, such that the roots are given by $\Delta = \Delta^+ \cup \Delta^-$ where Δ^+ and Δ^- are respectively the sets of positive and negative roots for the lexicographic order, i.e., $\Delta^- = -\Delta^+$ and

$$\Delta^+ = \{\varepsilon_j \pm \varepsilon_k, 1 \leq j < k \leq r\} \cup \{\varepsilon_j, 1 \leq j \leq r\}^{N-2r},$$

---

[2] We use the same notation D for the corresponding representations of $\mathscr{U}$ and $\mathfrak{so}(2,N-2)$.

where $E^0 = \emptyset$ and $E^1 = E$ for any set E. A more appropriate basis of $\mathfrak{g}^{\mathbb{C}}$ is given by the family $(X_{ij})_{-r \le i,j \le r}$, such that:

$$X_{ij} = -X_{ji} \text{ for } -r \le i,j \le r, \quad H_j = X_{-j,j} \text{ for } 1 \le j \le r,$$

and satisfying the following commutation relations:

$$[H, \pm X_{\pm j, \pm\sigma k}] = \pm(\varepsilon_j + \sigma\varepsilon_k)(H)(\pm X_{\pm j,\pm\sigma k}), \quad \forall H \in \mathfrak{h}, \tag{2.8}$$

$$[X_{j,\sigma k}, -X_{-j,-\sigma k}] = H_j + \sigma H_k, \tag{2.9}$$

where $|j| \neq |\sigma k|$, $1 \le j,k \le r$, and

$$\sigma \in \begin{cases} \{-1,1\} \text{ if } N \text{ is even,} \\ \{-1,0,1\} \text{ if } N \text{ is odd.} \end{cases}$$

In this way one sees that when $i,j \ge 1$, X_{ij} corresponds to the root $\varepsilon_i + \varepsilon_j$, $X_{-i,j}$ to the root $-(\varepsilon_i - \varepsilon_j)$, and so on. The set Δ_c^+ of positive compact roots is obtained by restricting the indices $i,j,\ldots$ of the roots to $\{-r,\ldots,r\} \setminus \{-1,1\}$. The remaining positive roots are the noncompact ones, the set of which we write Δ_n^+.

Now let D be a singleton representation of G realized on a Banach space $\mathscr{H}$. Then one can write, under the action of $\mathfrak{k}$:

$$\mathscr{H}^{\infty} = \oplus_{\mu} K(\mu), \tag{2.10}$$

each $K(\mu)$ being an irreducible $\mathfrak{k}$-module. The action of $\mathfrak{p}^{\pm} = \mathfrak{p}^{\mathbb{C}} \cap \mathfrak{n}^{\pm}$, which is generated by the family $(X_{\pm 1,j})_{|j| \neq 1}$, sends a nonzero $\mathfrak{k}$-module $K(\mu_1, \vec{\mu})$ to an irreducible one $K(\mu_1 \pm 1, \vec{\mu}')$ for some $\vec{\mu}'$, since D is a singleton. Suppose that $K(\mu) \neq \{0\}$ but $X_{\pm 1,2}K(\mu) = \{0\}$. Then the second component of the weights of $\mathscr{U}(\mathfrak{p}^{\pm})K(\mu)$ are bounded from above by μ_2. Two cases arise: $N > 5$ or $N = 4$ (since $n \ge 3$), the first one being the only one for which $\mathfrak{k}$ is semi-simple. Let $N \ge 5$. What precedes implies that $\mathscr{U}(\mathfrak{p}^{\pm})K(\mu)$ is finite dimensional because each weight is a Δ_c^+-dominant integer. Indeed, some power of $X_{\pm 1,j}$, $|j| \notin \{0,1\}$, is zero on $K(\mu)$. If N is odd, some power of $X_{\pm 1,0}$ is also vanishing, thanks to the relation $[X_{20}^m, X_{\pm,-2}^m] = m!X_{\pm,0}^m$.

Thus for an infinite-dimensional D, there exists $\varepsilon \in \{-1,1\}$ such that the powers of $X_{\varepsilon,2}$ are not vanishing on nonzero $\mathfrak{k}$-modules. Without loss of generality, we shall consider from now on that $\varepsilon = -1$, i.e., $X_{-1,2}K(\mu) \neq \{0\}$ for each nonzero $K(\mu)$. It follows that $D|_{\overline{K}_n} = \oplus_{l \in \mathbb{Z}} \pi_{\lambda + l(\varepsilon_1 - \varepsilon_2)}$, hence $D|_{\overline{K}_n}$ is a highest weight representation, i.e., there exist a weight $\lambda^{(0)}$ such that $D|_{\overline{K}_n} = \oplus_{l \in \mathbb{N}} \pi_{\lambda^{(0)} - l(\varepsilon_1 - \varepsilon_2)}$. We shall write $D_{\lambda^{(0)}}$ or $D(-\lambda_1^{(0)}, \vec{\lambda^{(0)}})$ for such a representation.

Let $K(\mu) \neq \{0\}$. Since $[\mathfrak{k}^{\mathbb{C}} \cap \mathfrak{n}^+, X_{-1,2}] = \{0\}$, $X_{-1,2}^2 K(\mu)$ is an irreducible $\mathfrak{k}$-module of weight $\mu - 2(\varepsilon_1 - \varepsilon_2) = (\mu_1 - 2, \mu_2 + 2, \mu_3, \ldots, \mu_r)$. The element

$Y_{-1,-1} = \sum_{|k|\neq 1} X_{-1,k}X_{-1,-k}$ of $\mathscr{U}^{\mathbb{C}}$ satisfies $[\mathfrak{k}^{\mathbb{C}}, Y_{-1,-1}] = \{0\}$, thus $Y_{-1,-1}K(\mu)$ is also an irreducible $\mathfrak{k}$-module of weight $\mu - 2\varepsilon_1 = (\mu_1 - 2, \mu_2, \mu_3, \dots, \mu_r)$. As D is an infinite-dimensional singleton, $X^2_{-1,2}K(\mu) \neq \{0\}$ and the multiplicity of $\mu_1 - 2$ is 1. Then one has necessarily $Y_{-1,-1}K(\mu) = \{0\}$. It follows that $Y_{-1,-1} = 0$ on $\mathscr{H}^\infty$. Finally the application of the adjoint representation on $Y_{-1,-1}$ yields the following relation on $\mathscr{H}^\infty$:

$$\sum_{i=-r}^{r} X_{j,-i}X_{-k,i} + \frac{N-2}{2}X_{-k,j} + \frac{2}{N}\delta_{jk}\mathbf{C}_2 = 0, \quad \forall j,k \in \{-r,\dots,r\}. \tag{2.11}$$

The $N = 4$ case is more simple since $\mathfrak{g}^{\mathbb{C}}$ is isomorphic to a direct sum of two copies of $\mathfrak{sl}(2)^{\mathbb{C}}$ and D is a singleton if and only if its restriction to one, and only one, of the two copies is trivial, but this property is equivalent to the relation (2.11).

Finally the fundamental relation (2.6) follows for D as a representation of $\mathfrak{g}$ and it has been proved in [4] that each nontrivial representation that satisfies (2.6) is a singleton (finite or infinite dimensional).

The following result is a characterization of the infinite-dimensional singletons of the n-anti-de Sitter group (or $(n-1)$-conformal group) $\overline{SO}_0(2,n-1)$.

Corollary 2.3 *Let D an infinite-dimensional IR of* $\overline{S}_n = \overline{SO}_0(2,n-1)$, $n \geq 3$. *The following conditions are equivalent:*

i) D is a singleton;

ii) The restriction $D|_{\overline{L}_n}$ *of D to the n-Lorentz group* $\overline{SO}_0(1,n-1)$ *is a UIR;*

iii) The restriction $D|_{\overline{P}_{n-1}}$ *of D to the* $(n-1)$*-Poincaré group* $\overline{SO}_0(1,n-2) \ltimes T_{n-1}$ *is a UIR.*

Proof. It has been proved in [4] that an irreducible infinite-dimensional representation which satisfies the fundamental relation (2.6) is irreducible when restricted to the n-Lorentz group. This proves the implication i) ⇒ ii). The implication iii) ⇒ i) has been proved in the same paper and the proof of ii) ⇒ iii) is in [25].

Remark 2.4 The restriction of an infinite-dimensional singleton of the n-anti-de Sitter group to the $(n-1)$-anti-de Sitter one (in other words one restricts from the conformal group to the anti-de Sitter one) is not irreducible in general. In fact there is only one case for which it is not irreducible, but it is a sum of two irreducible ones, as shown in theorem 3.4.

Let U a nontrivial UIR of the $(n-1)$-Poincaré group $\overline{P}_{n-1}$, the invariance group of the $(n-1)$-dimensional Minkowski space M_{n-1}. Then it is proved in [4] that if U extends to $\overline{S}_n = \overline{G}_{n-1}$, the conformal group of M_{n-1}, then the extension is a singleton uniquely defined by U.

Let D a finite-dimensional singleton of $\overline{SO}(N)$, $N \geq 4$. Then the restriction of D to $\overline{SO}(N-1)$ is irreducible if and only if N is even. If it is odd, the restriction is a sum of two irreducible ones.

2.2. Classification of the singletons

The following result is a corollary of the theorem 2.2. We write again D_μ or $D(-\mu_1,\vec{\mu})$ for a highest weight irreducible representation with weight $(\mu_1,\vec{\mu})$, of the anti-de Sitter group. The corresponding common Cartan subalgebra of $\mathscr{S}_n$ and of its maximal compact subalgebra $\mathfrak{k}_n$ is the one introduced in the preceding section. An infinite-dimensional highest weight representation is thus, with this choice, a positive minimal energy representation, $-\mu_1$ being the energy.

Theorem 2.5 *Let D a singleton representation of* $G = \overline{SO}(2,N-2)$. *Then, for a certain order of the roots, D is a highest weight IR such that:*

If D is finite dimensional, D is equivalent to one of the following series:

$$D(-\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}), \qquad \text{if } N \text{ is odd,} \tag{2.12}$$

$$D(-s,s,\dots,s,\varepsilon s), \qquad 2s \in \mathbb{N}\setminus\{0\} \text{ and } |\varepsilon| = 1, \text{ if } N \text{ is even.} \tag{2.13}$$

If D is infinite-dimensional, then it is a unitary representation equivalent to one of the following series:

$$D(s+\frac{N-4}{2},s,\dots,s), \qquad s \in \{0,\frac{1}{2}\}, \text{ if } N \text{ is odd,} \tag{2.14}$$

$$D(s+\frac{N-4}{2},s,\dots,s,\sigma s), \qquad 2s \in \mathbb{N} \text{ and } |\sigma| = 1, \text{ if } N \text{ is even.} \tag{2.15}$$

Proof. Assume $N \geq 5$. Since D is a weight representation, one can assume that it is a HW one, with weight $\lambda^{(0)}$. Let $v \in \mathscr{H}^\infty$ be a maximal vector, i.e., such that $\mathfrak{n}^+ v = \{0\}$ and $Hv = \lambda^{(0)}(H)v$ for all $H \in \mathfrak{h}$. Then applying the fundamental relation (2.11) to v yields

$$(\lambda_i^{(0)} - \lambda_{i+1}^{(0)})(\lambda_i^{(0)} + \lambda_{i+1}^{(0)} + \frac{N-4}{2} + 1 - i) = 0, \text{ where } 1 \leq i \leq r-1, \tag{2.16}$$

and, if N is odd,

$$\sum_{i=1}^{r} \lambda_i^{(0)} = -\frac{2}{N}\mathbf{C}_2. \tag{2.17}$$

Now writing $s = \lambda_2^{(0)}$, (2.16) gives the desired result when N is even, no matter if D is finite dimensional or not. If N is odd one has also, thanks to (2.17), $s(s-1/2) = 0$.

If $N = 4$, then one has $\lambda^{(0)} = (\lambda_1^{(0)},\lambda_2^{(0)})$ and thanks to the fundamental relation one gets $|\lambda_1^{(0)}| = |\lambda_2^{(0)}|$, which means that D is trivial on one of the two copies of $\mathfrak{so}(3)^{\mathbb{C}}$ (see the Remark below).

Remark 2.6 Let $\mathbf{J}_2$ be the (second order) Casimir operator of the "spin" subalgebra $\mathfrak{so}(N-2)$ of $\mathfrak{g}$. Then the relation (2.6) is equivalent to

$$\mathbf{J}_2 - H_1^2 = \frac{N-4}{N}\mathbf{C}_2.$$

In particular, if $N=4$ one has $H_1^2 = H_2^2$, since $\mathbf{J}_2 = H_2^2$.

On the lowest energy level of the singleton D the corresponding representation of $\mathfrak{so}(N-2)$ is itself a (finite-dimensional) singleton. If, conversely, one starts with a singleton representation of $\mathfrak{so}(N-2)$, then this can be the ground state of two singleton representations of $\mathfrak{g}$: one being finite dimensional, the other infinite dimensional.

2.3. A remarkable property of singletons

Here we present a generalization of the theorem of Flato and Frønsdal [13]: "*singleton* $\otimes$ *singleton* $= \oplus$ *massless representations*" proved in 1978 for the case $N=4$.

Let $\mathrm{Rac} = D(\frac{N-4}{2},0,\dots,0)$ and $\mathrm{Di}^{\pm} = D(\frac{1}{2}+\frac{N-4}{2},\frac{1}{2},\dots,\frac{1}{2},\varepsilon\frac{1}{2})$, where $|\varepsilon|=1$ and $\pm$ is the sign of ε. Then the former is the spin 0 unitary singleton while the latter is one of the two unitary singletons (resp. the unitary singleton) with spin $\frac{1}{2}$ if N is even (resp. odd). If N is odd, $\varepsilon=1$. We just write Di when $\varepsilon=1$, N odd or even.

Theorem 2.7

$$\mathrm{Rac}\otimes\mathrm{Rac} = \oplus_{s=0}^{\infty} D(s+N-4,s,0,\dots,0), \tag{2.18}$$

$$\mathrm{Rac}\otimes\mathrm{Di}^{\pm} = \oplus_{s-\frac{1}{2}=0}^{\infty} D(s+N-4,s,\frac{1}{2},\dots,\frac{1}{2},\varepsilon\frac{1}{2}). \tag{2.19}$$

Proof. Let D' and D'' two unitary singletons of $\overline{G}$, both with the same energy sign, i.e., both highest weight representations or both lowest ones. We are interested in reducing the product $D'\otimes D''$. Thus if D is an irreducible representation contained in this product then $D(M_{ab}) = D'(M_{ab})\otimes 1 + 1\otimes D''(M_{ab})$ for all a and b. For simplicity of the proof we use the notation introduced just before Theorem 2.2 and we write M', M'', and M instead of $D'(M)\otimes 1$, $1\otimes D''(M)$, and $D(M)$, respectively. Then one has, since D' and D'' satisfy the fundamental relation (2.6),

$$\begin{aligned} M'^2 &= \frac{N-2}{2}M' + \frac{2}{N}\mathbf{C}'_2\delta, \\ M''^2 &= \frac{N-2}{2}M'' + \frac{2}{N}\mathbf{C}''_2\delta. \end{aligned} \tag{2.20}$$

Then one gets, because of the relation $M^2 = M'^2 + M'M'' + M''M' + M''^2$:

$$M^2 = \frac{N-2}{2}M + \frac{2}{N}(\mathbf{C}'_2+\mathbf{C}''_2)\delta + K^2, \tag{2.21}$$

where $K^2 = M'M'' + M''M'$. More generally we define K^k, $k \geq 2$, by

$$K^k = \overbrace{M'M''M'M''\cdots}^{k\text{ terms}} + \overbrace{M''M'M''M'\cdots}^{k\text{ terms}}.$$

Multiplying both sides of (2.21) by M and using (2.20), one finds

$$M^3 = \frac{N-2}{2}M^2 + \frac{2}{N}(\mathbf{C}'_2 + \mathbf{C}''_2)M + \frac{N-2}{2}K^2 + \frac{2}{N}(\mathbf{C}'_2 M'' + \mathbf{C}''_2 M') + K^3. \tag{2.22}$$

Let $D' = D'' = \text{Rac}$. Then $\mathbf{C}'_2 = \mathbf{C}''_2 = -\frac{N(N-4)}{4}$, if one identifies the Casimir operators with their scalar values. It follows that $\mathbf{C}'_2 M'' + \mathbf{C}''_2 M' = -\frac{N(N-4)}{4}M$. The expression of K^2 follows from (2.21). It remains to calculate K^3. For let us define the symmetrizer S by $S(M_{ab}M_{cd}) = \frac{1}{2}(M_{ab}M_{cd} + M_{cd}M_{ab})$ and consider the element of the enveloping algebra $\Lambda^2_{abcd} = S(M_{ab}M_{cd}) + S(M_{bc}M_{ad}) + S(M_{ca}M_{bd})$. Then Λ^2_{abcd} is completely skew-symmetric in the indices a, b, c, d and is sent to zero by Rac, what we may write $\Lambda'^2_{abcd} = 0$. This implies, for each a, d:

$$\begin{aligned}\sum_{bc}\Lambda'^2_{abcd}M''^{bc} = & 2\sum_{bc}M'_{ab}M''^{bc}M'_{cd} - \sum_{bc}M'_{ab}M''^{bc}\eta_{cd} \\ & + \sum_{bc}M'_{bc}M''^{bc}(M'_{ad} - \eta_{cd}) = 0,\end{aligned} \tag{2.23}$$

hence $M'M''M' = M'M'' + \frac{1}{2}\text{Tr}(M'M'')(M' - \delta)$. A similar formula holds for $M''M'M''$. Thus, after adding them, one gets

$$K^3 = K^2 + \frac{1}{2}\text{Tr}(M'M'')(M - 2\delta).$$

Since $\mathbf{C}_2 = \frac{1}{2}\text{Tr}(M^2) = \frac{1}{2}\text{Tr}(M'^2 + M'M'' + M''M' + M''^2)$ one has $\text{Tr}(M'M'') = \mathbf{C}_2 - (\mathbf{C}'_2 + \mathbf{C}''_2)$ thanks to the relations above and, after factorizing, it follows that

$$[M^2 - (N-3)M - \frac{1}{2}\mathbf{C}_2\delta](M - 2\delta) = 0. \tag{2.24}$$

Let us determine explicitly the corresponding representations D. Since Rac is a highest weight representation, D is a also a highest weight one, say λ. The lowest energy of Rac being $\frac{N-4}{2}$, one has necessarily $\lambda_1 \leq -(N-4)$. To determine the possible values of λ, let us complexify $\mathfrak{g}$ and consider the (X_{ij})-basis version of the relation (2.24):

$$[X^2 + (N-3)X - \frac{2}{2}\mathbf{C}_2\delta](X + 2\delta) = 0. \tag{2.25}$$

Then the application of this relation on the maximal vector (ground state) v_λ of D yields:

$$[\lambda_1^2 + (N-3)\lambda_1 - \frac{1}{2}\mathbf{C}_2](\lambda_1 + 2) = 0.$$

Since $\lambda_1 \leq -(N-4)$, if $N \geq 5$ it follows:

$$\lambda_1^2 + (N-3)\lambda_1 - \frac{1}{2}\mathbf{C}_2 = 0. \tag{2.26}$$

If $N = 4$, we shall see that $\lambda_1 = -2$ is a solution of Equation (2.26). Thus we can work only with the latter. For the second component of λ one gets, since this weight is a Δ_c^+-dominant integer:

$$\lambda_2^2 + \lambda_1 + (N-4)\lambda_2 - \frac{1}{2}\mathbf{C}_2 = 0. \tag{2.27}$$

Subtracting this equation from the preceding one gives:

$$(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 + N - 4) = 0,$$

from which follows $\lambda_1 = -(\lambda_2 + N - 4)$. This condition implies that the other components are zero, i.e., $\lambda = (-s-N+4, s, 0, \ldots, 0)$ where $s = \lambda_2$. Then the integrability conditions imply that s is an integer. The necessary condition is thus proved. To prove that this is sufficient consider the following vectors, where $s \in \mathbb{N}$, v' is the maximal vector of D' and v'' the maximal one of D'' (see Remark 2.8 below):

$$v_s = \sum_{l=0}^{s} \frac{s!(-1)^l}{(s-l)!l!} \frac{\Gamma(-\frac{N-4}{2} - l + 1)\Gamma(-\frac{N-4}{2} - s - l + 1)}{\Gamma(-\frac{N-4}{2} + 1)^2} (X_{-1,2}^{s-l} v') \otimes (X_{-1,2}^{l} v''). \tag{2.28}$$

Then $\mathscr{U}(\mathfrak{g})v_s$ carries an irreducible highest weight representation of $\mathfrak{g}$ with weight $\lambda = (-s-N+4, s, 0, \ldots, 0)$. This finishes the proof for the Rac $\otimes$ Rac part.

To prove the other part let again $D' = \text{Rac}$ but $D'' = \text{Di}$. Then $\mathbf{C}''_2 = -\frac{N(N-3)}{8} \neq \mathbf{C}'_2$ and Λ''^2_{abcd} is no longer zero. But considering the fourth degree of M and using the relations (2.21) and (2.22) one gets:

$$\begin{aligned} M^4 =& (N-2)M^3 + \Big[-\Big(\frac{N-2}{2}\Big)^2 + \frac{2}{N}(\mathbf{C}'_2 + \mathbf{C}''_2) \Big] M^2 \\ & - (N-2)\frac{2}{N}(\mathbf{C}'_2 + \mathbf{C}''_2)M - \Big(\frac{2}{N}\Big)^2 (\mathbf{C}'_2 + \mathbf{C}''_2)^2 \delta + (K^2)^2. \end{aligned} \tag{2.29}$$

After some calculations, we find

$$(K^2)^2 = K^4 + \frac{N-2}{2}\Big[K^3 + \frac{2}{N}(\mathbf{C}'_2 M'' + \mathbf{C}''_2 M')\Big] + \frac{8}{N}\mathbf{C}'_2 \mathbf{C}''_2 \delta.$$

From Equation (2.22) one may write $K^3 + \frac{2}{N}(\mathbf{C}'_2 M'' + \mathbf{C}''_2 M')$ in terms of M, M^2, and M^3. Thus it remains to calculate K^4. After lengthy calculations one arrives to

$$S(K^4) = [N - 2 + \mathrm{Tr}(M'M'')]\Big[M^2 - \frac{N-2}{2}M - \frac{2}{N}(\mathbf{C}'_2 + \mathbf{C}''_2)\delta\Big].$$

This relation, together with the others above yields after other lengthy calculations an expression for $S(M^4)$ from which one gets the factorized relation:

$$[M^2-(N-3)M-\frac{1}{2}(\mathbf{C}_2-\frac{(N-4)(N-5)}{8})\delta]\,(M-\frac{N-1}{2}\delta)\,(M-\frac{3}{2}\delta)=0. \tag{2.30}$$

Again, in order to identify explicitly the representations contained in $\mathrm{Rac}\otimes\mathrm{Di}$, we write the preceding relation in terms of the basis (X_{ij}) of $\mathfrak{g}^{\mathbb{C}}$. Then one gets:

$$[X^2+(N-3)X-\frac{1}{2}(\mathbf{C}_2-\frac{(N-4)(N-5)}{8})\delta]\,(X+\frac{N-1}{2}\delta)(X+\frac{3}{2}\delta)=0. \tag{2.31}$$

Using similar arguments as for the case $\mathrm{Rac}\otimes\mathrm{Rac}$ yields

$$\lambda_1^2+(N-3)\lambda_1-\frac{1}{2}(\mathbf{C}_2-\frac{(N-4)(N-5)}{8})=0, \tag{2.32}$$

and, for λ_2, the weight being a Δ_c^+-dominant integer:

$$\lambda_2^2+\lambda_1+(N-4)\lambda_2-\frac{1}{2}(\mathbf{C}_2-\frac{(N-4)(N-5)}{8})=0. \tag{2.33}$$

After subtracting this equation from the preceding one it follows:

$$(\lambda_1-\lambda_2)(\lambda_1+\lambda_2+N-4)=0,$$

from which finally one gets $\lambda=(-s-N+4,s,\frac{1}{2},\dots,\frac{1}{2})$, where $s=\lambda_2$ and $s-\frac{1}{2}\in\mathbb{N}$. Now to obtain the sufficient condition we consider again the vectors v_s of (2.28) but with some changes: We replace in the right-hand side s by $s-\frac{1}{2}$. Evidently v'' is now a maximal vector for Di. Then again $\mathscr{U}(\mathfrak{g})v_s$ carries an irreducible highest weight representation of $\mathfrak{g}$, but now with weight $\lambda=(-s-N+4,s,\frac{1}{2},\dots,\frac{1}{2})$.

The case $D''=\mathrm{Di}^-$ is similar. The only change is $\lambda_r=-\frac{1}{2}$ instead of $\lambda_r=\frac{1}{2}$.

Remark 2.8 One may reduce the product $\mathrm{Di}\otimes\mathrm{Di}$ in the same manner as in Theorem 2.7, i.e., by seeking for an ideal which is sent to zero by each representation contained in $\mathrm{Di}\otimes\mathrm{Di}$. Another way to reduce this product is to consider one of the Di's as a summand in the product $\Sigma\otimes\mathrm{Rac}$, Σ being the spinor representation, and consider the product $\Sigma\otimes\mathrm{Rac}\otimes\mathrm{Di}$.

The vectors v_s appearing in (2.28) can be realized explicitly in a very simple manner. For this, let us realize the Rac on the cone $\{y\in\mathbb{R}^{2,N-2}\mid y^2=0\}$ in the usual way. Then one may choose v' to be the function $y\mapsto(x_1)^{-\frac{N-4}{2}}$ where $x_1=\frac{\sqrt{-1}y_{1}-y_0}{\sqrt{2}}$. For the other Rac, v'' is defined in a similar manner but with primes on the variables. For the Di, if v'' is the maximal vector, one can choose the map $y'\mapsto(x'_1)^{-\frac{N-3}{2}}w_{\frac{1}{2}}$, where $w_{\frac{1}{2}}$ is the maximal vector of Σ. Now let $\sigma=0$ or $\frac{1}{2}$,

$w_0 = 1$, and define $x_2 = \frac{y_1+\sqrt{-1}y_2}{\sqrt{2}}$ and x_2' by a similar formula, but with primes on the variables. D'' can be either a Rac or a Di. Then the map

$$v_s : (y, y') \mapsto (x_1x_1')^{-\frac{N-4}{2}-s}(x_1x_2' - x_2x_1')^{s-\sigma} w_\sigma,$$

where $s - \sigma \in \mathbb{N}$, is a maximal vector of the representation $D(s+N-4, s, \sigma, \dots, \sigma)$.

3. Masslessness

Since the anti-de Sitter space-time H_n^ρ contracts to the Minkowski space-time M_n, it is reasonable to demand that the notion of masslessness on H_n^ρ should correspond under the contraction to masslessness on M_n, where the square of the mass operator (usually denoted by $\sum P_\mu P^\mu$) is sent to 0 by massless representations of the Poincaré group $\overline{P}_n$. Such representations, when they do not have continuous spin (we will say *discrete spin*)[3], are induced by a unitary finite dimensional representation of a semi-direct product of the Euclidean subgroup $\overline{E}(n-2)$ (contained in $\overline{L}_n$) by the subgroup of translations T_n. Their restrictions to the translations T_{n-2} of $\overline{E}(n-2)$ are trivial. The subgroup $\overline{E}(n-2)$ is such that the isotropic cone of M_n is homeomorphic to $\overline{L}_n/\overline{E}(n-2)$. Contractibility may be used as a criterion for masslessness on the anti-de Sitter space. In the definition we will give below, by *natural contraction* of a representation of the anti-de Sitter group to a representation of the Poincaré group shall mean a contraction which leaves the restriction to the Lorentz group L_n (contained in both of them) invariant up to equivalence. It is thus compatible with the contraction $H_n^\rho \xrightarrow{\rho\to 0} M_n$. For example, the minimal energy representation $U = D(E_0, \vec{\lambda})$ may be contracted to a representation of the Poincaré group. In terms of the curvature, $\sqrt{\rho}E_0$ is sent to the mass. If E_0 is fixed, for example equal to $\frac{n-3}{2}$, then the resulting mass is 0. But the contracted representation is massless for the Poincaré group if it is not trivial on the translations.

Definition 3.1 *A unitary representation U of the anti-de Sitter group $\overline{S}_n$ is said to be* massless *if U contracts naturally to a discrete spin massless representation of the Poincaré group $\overline{P}_n$.*

An immediate consequence is that singletons of $\overline{S}_n$ are not massless representations. Indeed such a singleton contracts to a representation of the Poincaré group $\overline{P}_n$ which is trivial when restricted to the subgroup of translations T_n [25]. But singletons are not massive particles either, in the sense that a massive particle on the anti-de Sitter space-time must be described by a representation of $\overline{S}_n$ which contracts to a massive representation of the Poincaré group $\overline{P}_n$, thus necessarily nontrivial on the translations. Hence singletons have no analog in Minkowski spaces M_n. This was already pointed out by Flato and Frønsdal for the $n = 4$ case (see for example [15]).

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[3] When the space-time dimension n is even helicity is easily defined: It is a straightforward generalization of the notion of helicity in the 4-dimensional case. Thus a discrete spin representation is nothing but discrete helicity representation when n is even (see Remark 7 in [25]).

The notion of masslessness is not unique since several nonequivalent representations of $\overline{S}_n$ may be contracted to a massless representation of $\overline{P}_n$. Below we shall consider two notions of masslessness, both closely related to singletons. To distinguish them we give the following Definitions.

Definition 3.2 *A massless representation U of the anti-de Sitter group $\overline{S}_n$ is said to be* conformal massless *if there exists a singleton D of the conformal group $\overline{G}_n$ such that $U \sim D|_{\overline{S}_n}$.*

This is the classical definition of masslessness given by Flato and Frønsdal when $n = 4$ (see for example [14]). A remarkable fact is that a conformal massless representation $U \sim D|_{\overline{S}_n}$ contracts naturally to the restriction $D|_{\overline{P}_n}$ of the singleton D to the Poincaré group. $D|_{\overline{P}_n}$ is irreducible (see Corollary 2.3) and is a discrete spin massless representation of $\overline{P}_n$ (see [4]).

It is important to notice that since the unitarity of U is required, only infinite-dimensional (unitary) singletons D may be used in the preceding definition.

Next, we present a second notion of masslessness. We shall see that it does not coincide with the first notion given, if $n \geq 5$.

Definition 3.3 *A massless representation U of the anti-de Sitter group $\overline{S}_n$ is said to be* composite massless *if U occurs in the reduction of the tensor product $D_1 \otimes D_2$ where D_1 and D_2 are irreducible weight representations, with the same energy sign, equivalent to a* Rac *or a* Di.

This definition means that a composite massless representation describing a massless particle on the anti-de Sitter space is composed of two subparticles, the singletons, in the same manner as nucleons are composed of quarks, except that singletons are unobservable for kinematical reasons while the unobservability of quarks is due to their confinement [15]. The representations appearing in the right-hand side of Theorem 2.7 are composite massless representations. They were considered for $n = 5$ by Ferrara and Frønsdal as the massless ones in [11, 12]. Irreducibility and Gupta-Bleuler (GB) quantization are almost always possible, but there are few exceptions. Some results concerning GB quantization are given in Section 4. Here we list the conformal massless representations.

Theorem 3.4. **([25])** *Let U be a conformal massless representation of the anti-de Sitter group $\overline{S}_n$. Then for a certain order of the roots:*

$$n \textit{ even} \Longrightarrow \begin{cases} U \sim D(s+\frac{n-2}{2}, s, \ldots, s), \textit{ for } s \neq 0 \textit{ such that } 2s \in \mathbb{N} \quad \textit{or} \\ U \sim D(\frac{n-2}{2}, 0, \ldots, 0) \oplus D(\frac{n}{2}, 0, \ldots, 0), \end{cases} \tag{3.1}$$

$$n \textit{ odd} \Longrightarrow \begin{cases} U \sim D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \oplus D(\frac{n-1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}) \quad \textit{or} \\ U \sim D(\frac{n-2}{2}, 0, \ldots, 0) \oplus D(\frac{n}{2}, 0, \ldots, 0). \end{cases} \tag{3.2}$$

In particular there is no conformal massless representations with spin different from 0 or $\frac{1}{2}$ in odd-dimensional anti-de Sitter spaces. It can be seen that this is still true in odd-dimensional Minkowski spaces if one defines conformal masslessness in a similar fashion, i.e., by restricting a singleton of the conformal group to the Poincaré group.

The following questions naturally arise: Are the conformal massless representations composite massless? What about the converse? The answer is practically negative. In fact, we have:

Theorem 3.5. **([25])** *A conformal massless representation is composite massless if and only if $n = 3$ or $n = 4$. More precisely, there is no tensor product of unitary weight representations with the same energy sign containing such a representation.*

Thus for $n \geq 5$, conformal invariance is not compatible with singleton composed of massless particles, provided that those singletons have the same energy sign. Though other type of composite particles are allowed. For example one can find conformal massless representations contained in the tensor product of a singleton by some multipleton or in the product of some two multipletons (see Remark 1 in [25]). Here we call a multipleton (or *m*-ton, for a certain *m*) a representation for which the diagram of maximal weights is included in *m* parallel lines: a 1-ton is a singleton, a 2-ton is a doubleton, and so on. Multipletons are generally not unitary; it is the case of the multipletons concerned by the "compositeness" of conformal massless representations.

4. Gupta-Bleuler quantization

For simplicity, we consider in this section that $n \geq 4$.

The method used to construct GB triplets is the following: Suppose that the IR $D(E, \vec{\lambda})$ is unitary if $E \geq E_0$ (E_0 is the *limit of unitarity*). Then usually when $E \to E_0$ ($E \neq E_0$), even for $E < E_0$, the IR $D(E, \vec{\lambda})$ becomes indecomposable. More precisely one obtains a (non-direct) sum of $D(E_0, \vec{\lambda})$ with another representation. The (physical) representation $D(E_0, \vec{\lambda})$ is realized as a quotient (by the gauge representation). Then using a third (scalar) representation together with some conditions usually satisfied one may construct a GB triplet.

The physical and gauge representations are usually minimal or maximal energy representations and are related to the Verma modules in the following way. Let $\mathfrak{g}$ a noncompact semi-simple Lie algebra, the typical example being $\mathfrak{so}(2, N-2)$. Let $\mathfrak{k}$ the (reductive) maximal compact subalgebra of $\mathfrak{g}$; $\mathfrak{k} \simeq \mathbb{R} \oplus \mathfrak{so}(N-2)$ for $\mathfrak{so}(2, N-2)$. Let $K(\lambda)$ a finite-dimensional simple $\mathfrak{k}$-module with weight λ. As usual the energy is $E = -\lambda_1$. Then there exists E_0 depending on $\vec{\lambda}$ such that the $\mathfrak{g}$-module $N(\lambda) = \mathscr{U}(\mathfrak{g}^{\mathbb{C}}) \otimes_{\mathscr{U}(\mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^{+})} K(\lambda)$ is not unitarizable if $E = |\lambda_1| < E_0$ [2, 9]. In particular, if $\mathfrak{g} = \mathfrak{so}(2, N-2)$, this module is unitarizable if and only if $E \geq E_0$, $E_0 = \lambda_2 + k$, where k a constant depending on $\vec{\lambda}$. $N(\lambda)$ is not always irreducible. If

$\mathfrak{g} = \mathfrak{so}(2, N-2)$, $N(\lambda)$ is irreducible (i.e., $N(\lambda) = L(\lambda)$) if and only if $E > E_0$. If $E = E_0$ then $N(\lambda)$ is not irreducible. It contains a maximal submodule I generated by relations similar to those of (2.6), the fundamental relation of singletons, or to those of (2.24) and (2.30), satisfied by the composite massless representations, obtained in Section 2. The irreducible quotient $L(\lambda) = N(\lambda)/I$ corresponds to the physical space and carries the representation $D(E_0, \vec{\lambda})$ while an irreducible quotient of I corresponds to the gauge space. To construct a GB triplet one needs a third space, endowed with an indefinite metric, and a representation conjugated to the gauge defined on a quotient space, the so-called scalar space. This is the method for constructing GB triplets which we shall call *natural* in the remaining.

Note that GB triplets do not require unitarity of the three representations; only the physical one has to be unitary.

In what follows we will give some examples of GB triplets for all singletons and almost all of the massless representations. Futher details can be find in [25].

The representations we shall consider are those of the anti-de Sitter group $\overline{S}_n$. The *Lorentz conditions*, i.e., the conditions which define the space $\mathscr{H}_2$ (the physical space is $\mathscr{H}_2/\mathscr{H}_3$), follow directly from the relations defining the submodule I (e.g., see (2.6)).

4.1. Singletons: spin 0

Let $\mathscr{H}_1^{(0)}$ the space of square-integrable (with respect to the Riemannian measure) positive energy solutions of $(\partial^2)^2\varphi = 0$ and $y \cdot \partial\varphi = -\frac{n-3}{2}\varphi$, $\mathscr{H}_2^{(0)} = \{\varphi \in \mathscr{H}_1^{(0)} \mid \partial^2\varphi = 0\}$ and $\mathscr{H}_3^{(0)} = \{\varphi \in \mathscr{H}_2^{(0)} \mid \varphi \text{ has the form } y^2\phi\}$, where $y \in H^+ = \cup_{\rho>0} H_n^\rho$, $\partial^2 = \sum \partial_a \partial^a$, and $y \cdot \partial = \sum y^a \partial_a$. Then these spaces realize the GB triplet:

$$D(\frac{n+1}{2}, 0, \ldots, 0) \rightarrow D(\frac{n-3}{2}, 0, \ldots, 0) \rightarrow D(\frac{n+1}{2}, 0, \ldots, 0),$$

i.e., $\mathscr{H}_1^{(0)}/\mathscr{H}_2^{(0)}$ and $\mathscr{H}_3^{(0)}$ (resp. scalar and gauge space) carry the irreducible representation $D(\frac{n+1}{2}, 0, \ldots, 0)$ while the quotient $\mathscr{H}_2^{(0)}/\mathscr{H}_3^{(0)}$ (physical space) carry the singleton $D(\frac{n-3}{2}, 0, \ldots, 0)$. Since $\lim_{y^2 \to 0} \varphi = 0$ if $\varphi \in \mathscr{H}_3^{(0)}$ one may realize the singletons in $\mathscr{H}_2^{(0)}$ by taking $\lim_{y^2 \to 0} \varphi(y)$ or, equivalently, at the "boundary" of the space-time, i.e., by considering $\lim_{R \to \infty} R^{\frac{n-3}{2}} \varphi(y)$, where $R = \sqrt{\sum_{a=1}^{n-1} (y^a)^2}$.

4.2. Singletons: spin 1/2

Let Σ be the spinor representation on the spinor module V_Σ (not irreducible if $n-1$ is even), (γ_a) the Dirac matrices such that $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ (if $n-1$ is odd then γ_{n-1} is a multiple of the product of the others). As usual $\not{y} = \sum y^a \gamma_a$, $\not{\partial} = \sum \partial^a \gamma_a$. To realize the GB triplet explicitly we reduce the tensor product $D(E, \vec{0}) \otimes \Sigma$, $E > \frac{n-1}{2}$, to realize the irreducible representation $D(E - \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and then we let $E \rightarrow \frac{n-1}{2}$ to get an indecomposable representation from which we realize the following GB triplet. The spinors we are using are the maps $\Psi : H^+ \rightarrow V_\Sigma$.

Let $\mathscr{H}_1^{(1/2)}$ be the space of spinors which are square-integrable positive energy solutions of $\partial^2\Psi = 0$, $y\cdot\partial\Psi = -\frac{n-2}{2}\Psi$ and $(\not{y}\not{\partial})^2\Psi = 0$. We shall write $\mathscr{H}_2^{(1/2)} = \{\Psi \in \mathscr{H}_1^{(1/2)} \mid \not{\partial}\Psi = 0\}$ and $\mathscr{H}_3^{(1/2)} = \{\Psi \in \mathscr{H}_2^{(1/2)} \mid \Psi$ has the form $\not{y}\Phi\}$. Let $\nu = 0$, if n is even, and 1 otherwise. Then these spaces realize the GB triplet:

$$D(\frac{n}{2},\frac{1}{2},\dots,\frac{1}{2},(-1)^{\nu}\frac{1}{2}) \to D(\frac{n-2}{2},\frac{1}{2},\dots,\frac{1}{2}) \to D(\frac{n}{2},\frac{1}{2},\dots,\frac{1}{2},(-1)^{\nu}\frac{1}{2}),$$

i.e., $\mathscr{H}_1^{(1/2)}/\mathscr{H}_2^{(1/2)}$ and $\mathscr{H}_3^{(1/2)}$ (resp. scalar and gauge space) carry the irreducible representation $D(\frac{n}{2},\frac{1}{2},\dots,\frac{1}{2},(-1)^{\nu}\frac{1}{2})$ while the quotient $\mathscr{H}_2^{(1/2)}/\mathscr{H}_3^{(1/2)}$ (physical space) carries the singleton $D(\frac{n-2}{2},1/2,\dots,1/2)$. Since $\lim_{y^2\to 0}\not{y}\Psi = 0$ whenever $\Psi \in \mathscr{H}_3^{(1/2)}$ one can realize the singletons in $\mathscr{H}_2^{(1/2)}$ by taking $\lim_{y^2\to 0}\not{y}\Psi(y)$ or, equivalently, at the "boundary" by $\lim_{R\to\infty}R^{\frac{n-1}{2}}\Psi(y)$.

4.3. Singletons: spin $s\geq 1$

Here $n-1$ is necessarily even. The tensor product we use is $D(E,\vec{0})\otimes(\Sigma)^{\otimes 2s}$, with E close to $\frac{n-3}{2}+2s$. The corresponding multispinors are the maps $\Psi : H^+ \to \Sigma^{\otimes 2s}$. Define $\gamma_a^{(t)}$ by $\gamma_a^{(t)}v_1\otimes\cdots v_t\cdots v_{2s} = v_1\otimes\cdots\gamma_a v_t\cdots v_{2s}$, $\not{y}^{(t)} = \sum y^a\gamma_a^{(t)}$, $\not{\partial}^{(t)} = \sum\partial^a\gamma_a^{(t)}$, τ_{ij} the transposition $i \leftrightarrow j$ and

$$\mathscr{Y} = \frac{1}{2s}\Big[\sum_{1\leq t\leq 2s}\tau_{(t,2s)}\Big]\not{y}^{(2s)}.$$

Let $\mathscr{H}_1^{(s)}$ be the space of spinors which are square-integrable and positive energy solutions of $\partial^2\Psi = 0$, $y\cdot\partial\Psi = (-\frac{n-3}{2}-2s)\Psi$, and $(\sum_{t=1}^{2s}\not{y}^{(t)}\not{\partial}^{(t)})^2\Psi = 0$. Let $\mathscr{H}_2^{(s)} = \{\Psi\in\mathscr{H}_1^{(s)} \mid \not{\partial}^{(t)}\Psi = 0, \forall t\}$ and $\mathscr{H}_3^{(s)} = \{\Psi\in\mathscr{H}_2^{(s)} \mid \Psi$ has the form $\mathscr{Y}\Phi\}$. Then these spaces realize the GB triplet:

$$D(\frac{n-1}{2}+s,s,\dots,s-1) \to D(\frac{n-3}{2}+s,s,\dots,s) \to D(\frac{n-1}{2}+s,s,\dots,s-1),$$

i.e., $\mathscr{H}_1^{(s)}/\mathscr{H}_2^{(s)}$ and $\mathscr{H}_3^{(s)}$ (resp. scalar and gauge space) carry the irreducible representation $D(\frac{n-1}{2}+s,s,\dots,s-1)$ while the quotient $\mathscr{H}_2^{(s)}/\mathscr{H}_3^{(s)}$ (physical space) carries the singleton $D(\frac{n-3}{2}+s,s,\dots,s)$. Since $\lim_{y^2\to 0}\sum_t\not{y}^{(t)}\Psi = 0$ whenever $\Psi\in\mathscr{H}_3^{(s)}$, as before one can realize the singletons in $\mathscr{H}_2^{(s)}$ as the limit $\lim_{y^2\to 0}\sum_t\not{y}^{(t)}\Psi(y)$ or, again, at the "boundary" as the limit $\lim_{R\to\infty}R^{(\frac{n-3}{2}+2s)}\Psi(y)$.

4.4. Conformal massless representations: spin $s\geq 1$

Define ε by $|\varepsilon| = 1$ if $n-1$ is even and $\varepsilon = 1$ if $n-1$ is odd. The limit of unitarity of the IR $D(E_0,s,\dots,s,\varepsilon s)$ is $E_0 = \frac{n-2}{2}+s$ if n is even (or $n-1$ is odd) and $s\geq 1$, $E_0 = \frac{n-3}{2}+s$ if not, i.e., n odd (or $n-1$ even) or $s\in\{0,\frac{1}{2}\}$. Thus the construction of natural GB triplets for conformal massless representations is only possible in even-dimensional space-time, i.e., $n-1$ odd, and for spin greater or equal to 1. So let $s \geq 1$ and n even. We use the same tensor product as in the preceding subsection, but with E close to $\frac{n-2}{2}+2s$.

Let $\mathscr{H}_1^{(s)}$ be the space of multispinors which are square-integrable positive energy solutions of $\partial^2\Psi = 0$, $y \cdot \partial\Psi = (-\frac{n-2}{2}-2s)\Psi$ and $(\sum_{t=1}^{2s} \not{y}^{(t)}\not{\partial}^{(t)})^2\Psi = 0$. Let $\mathscr{H}_2^{(s)} = \{\Psi \in \mathscr{H}_1^{(s)} \mid \not{\partial}^{(t)}\Psi = 0,\ \forall\,\mathrm{t}\}$, $\mathscr{H}_3^{(s)} = \{\Psi \in \mathscr{H}_2^{(s)} \mid \Psi \text{ has the form } \mathscr{Y}\psi\}$, where

$$\mathscr{Y} = \frac{1}{2s(2s-1)}\Big[\sum_{1\leq t\leq 2s-1} \tau_{(t,2s-1)} + \sum_{1\leq t<t'\leq 2s-1} \tau_{(t,2s)}\tau_{(t',2s-1)}\Big]\big[\not{y}^{(2s-1)} - \not{y}^{(2s)}\big].$$

Then these spaces realize the GB triplet:

$$D(\frac{n-1}{2}+s,s,\ldots,s-1) \to D(\frac{n-3}{2}+s,s,\ldots,s) \to D(\frac{n-1}{2}+s,s,\ldots,s-1),$$

i.e., $\mathscr{H}_1^{(s)}/\mathscr{H}_2^{(s)}$ and $\mathscr{H}_3^{(s)}$ (resp. scalar and gauge space) carry the irreducible representation $D(\frac{n-1}{2}+s,s,\ldots,s-1)$ while the quotient $\mathscr{H}_2^{(s)}/\mathscr{H}_3^{(s)}$ (physical space) carries the conformal massless one $D(\frac{n-3}{2}+s,s,\ldots,s)$.

4.5. Composite massless representations: spin $s\geq 1$

If ε is defined in the same way as in the preceding subsection and if σ is the fractional part of s, then the limit of unitarity of the IR $D(E_0,s,\sigma,\ldots,\sigma,\varepsilon\sigma)$, is $n-3+s$ if $s \geq 1$, and $\frac{n-3}{2}+s$ otherwise. Thus natural GB triplets for the composite massless representations of this form can be obtained only for $s \geq 1$, regardless to the parity of n. So let $s \geq 1$. Then the corresponding GB triplets are realized in a somewhat known fashion (see for example [6, 7, 20, 22]).

First suppose that $s \in \mathbb{N}$. We reduce the tensor product $D(E,\vec{0}) \otimes D(-s,\vec{0})$, with E close to $s+n-3$. $D(-s,\vec{0})$ is a finite-dimensional representation realized on the space of polynomials in the variables $z_{-1},\ldots,z_{n-1}$. The GB triplet is realized on the space of functions $(y,z) \mapsto \phi(y,z)$ with the usual conditions on the variable $y \in H^+$. This is equivalent to realizing the representation on the space of symmetric tensor fields of rank s on H^+. The generators of the Lie algebra are $M_{ab} = y_a\partial_b - y_b\partial_a + z_a\delta_b - z_b\delta_a$, $\delta_c = \frac{\partial}{\partial z^c}$. Once the irreducible representation $D(E,s,0,\ldots,0)$ is realized for $E \neq s+n-3$, then one obtains an indecomposable representation after taking the limit $E \to s+n-3$. From there one constructs the GB triplet:

$$D(s+n-2,s-1,0,\ldots,0) \to D(s+n-3,s,0,\ldots,0) \to D(s+n-2,s-1,0,\ldots,0).$$

As above one needs some Lorentz conditions to fix the space $\mathscr{H}'^{(s)}_2$ which defines the physical situation: Its elements are the tensor fields φ which satisfy $\partial^2\varphi(y,z) = 0$, $y \cdot \partial\varphi(y,z) = -(s+n-3)\varphi(y,z)$ (homogeneity), $z \cdot \delta\varphi(y,z) = s\varphi(y,z)$, $\delta^2\varphi(y,z) = 0$ (φ is traceless), $\partial \cdot \delta\varphi(y,z) = 0$ (φ is divergenceless) and $y \cdot \delta\varphi(y,z) = 0$ (φ is transverse). The physical representation $D(s+n-3,s,0,\ldots,0)$

is realized on the quotient $\mathscr{H}'^{(s)}_2/\mathscr{H}'^{(s)}_3$ where the gauge space $\mathscr{H}'^{(s)}_3$ is the subspace of elements $\varphi \in \mathscr{H}'^{(s)}_2$ of the form $\varphi(y,z) = \left[y^2 z\cdot\partial + (n-3+2s)y\cdot z\right]\phi(y,z)$.

Now suppose that s is a half-integer, i.e., $s-\frac{1}{2} \in \mathbb{N}$. The tensor product we consider is $D(E,\vec{0}) \otimes D(-(s-\frac{1}{2}),\vec{0}) \otimes \Sigma$, with the same material as above but with $s-\frac{1}{2}$ instead of s for the first two representations. The desired representations act on tensor-spinor fields Ψ on H^+. The generators of the Lie algebra are $M_{ab} = y_a\partial_b - y_b\partial_a + z_a\delta_b - z_b\delta_a + \frac{1}{4}[\gamma_a,\gamma_b]$. To get the needed indecomposable representation we let $E \to s+n-3+\frac{1}{2}$. Then one gets the GB triplet:

$$D(s+n-2,s-1,\frac{1}{2},\ldots,\varepsilon\frac{1}{2}) \to D(s+n-3,s,\frac{1}{2},\ldots,\varepsilon\frac{1}{2}) \to$$
$$\to D(s+n-2,s-1,\frac{1}{2},\ldots,\varepsilon\frac{1}{2}).$$

The value of ε, ± 1, depends as usual on the parity of n and on the irreducible component of Σ used (Σ is irreducible only if $n-1$ is odd).

$\mathscr{H}'^{(s)}_2$ is defined from the Lorentz conditions: $\partial^2\Psi(y,z) = 0$, $y\cdot\partial\Psi(y,z) = -(s+n-3+\frac{1}{2})\Psi(y,z)$, $z\cdot\delta\Psi(y,z) = (s-\frac{1}{2})\Psi(y,z)$, $\delta^2\Psi(y,z) = 0$, $\partial\cdot\delta\Psi(y,z) = 0$, $y\cdot\delta\Psi(y,z) = 0$, $\not{y}\not{\partial}\Psi(y,z) = 0$, and $\not{z}\not{\delta}\Psi(y,z) = 0$. Finally the physical representation $D(s+n-3,s,\frac{1}{2},\ldots,\varepsilon\frac{1}{2})$ is realized on the quotient $\mathscr{H}'^{(s)}_2/\mathscr{H}'^{(s)}_3$ where the gauge space $\mathscr{H}'^{(s)}_3$ is the subspace of elements $\Psi \in \mathscr{H}'^{(s)}_2$ of the form $\Psi(y,z) = \left[y^2 z\cdot\partial + (n-3+2s-1)y\cdot z + \not{y}\not{z}\right]\Phi(y,z)$.

5. The unreasonable effectiveness of the 4-dimensional space-time

From what precedes one sees that singletons of the anti-de Sitter group are well defined for $n \geq 3$. They are defined for all half-integer spin if the space-time dimension n is odd and only for spin 0 or 1/2 if n is even. Moreover singleton theory is always quantizable in the sense of Gupta and Bleuler and the resulting gauge theory is topological in the sense that singletons appear at spatial infinity, since in the Lorentz condition one lets the "radius" R of space-time tend to infinity. Apparently there is no noticeable difference with the $n = 4$ dimensional case.

However masslessness behaves differently. Indeed if one needs conformal invariance one has to work with conformal masslessness, but then the space-time dimension must be even if massless particles with spins other than 0 or 1/2 are necessary. Furthermore the corresponding massless particles cannot be composed of subparticles like singletons if $n > 4$. Obviously the conformal invariance of masslessness is not always needed, in which case there is in general no unique way to define the masslessness notion, even in n-dimensional Minkowski space-time. This is due to the fact that if n is sufficiently large, the rank of the maximal compact subalgebra $\mathfrak{k}$ is greater than 2, thus there is no unique way to choose the right spinor part of the weight of the representation. But if the composite aspect of singletons is important, as they are in some theories, then composite masslessness becomes more appropriate, since for every spin and for every parity of the space-time dimension composite massless representations occur in the reduction of the tensor product of two singletons, i.e., the corresponding massless particles are composed of two singletons. Unfortunately the two notions of masslessness we have at hand are not compatible if $n > 4$: A physical theory on the anti-de Sitter space-time such that masslessness is conformal invariant and composite (the particle-anti-particle case being excluded) does not exist for $n > 4$.

Thus the $n = 4$ anti-de Sitter space-time appears to be the only one for which masslessness is well defined and is conformal invariant and composite, if one considers that an n-dimensional physical space-time satisfies necessarily $n \geq 4$.

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Remarks on the zeta-regularized determinant of differential operators

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Abstract. It is shown that the equalities $\det(\not{D}+im)=\det(\not{D}-im)=\sqrt{\det(\not{D}^2+m^2)}$, where $\not{D}$ is the massless Dirac operator, hold if and only if: (i) $\det(\not{D}+im)$ and $\det(\not{D}-im)$ are defined by using the symmetry property of the spectrum of $\not{D}$; (ii) $Z(\not{D}^2+m^2,0)$ is an even integer (the sign of the square root above is positive if $Z(\not{D}^2+m^2,0)$ is divisible by 4, and negative otherwise). On the other hand, the equality $|\det(\not{D}\pm im)|=\sqrt{\det(\not{D}^2+m^2)}$ is always holding. It is also shown, by applying the standard definition of zeta-determinant, that $\det(\not{D}+im)$ is not equal to $\det(\not{D}-im)$ in general. In order to show this fact, a variational formula for $\det(D+m)$ with respect to m is derived for general self-adjoint operators.

Keywords: zeta-function regularization, functional determinant

Mathematics Subject Classifications (2000): 81T16, 58J52

Introduction

Some doubts have been thrown on the validity of the following equalities

$$\det(\not{D}+im)=\det(\not{D}-im)=\sqrt{\det(\not{D}^2+m^2)} \tag{1}$$

(see, e.g. [6]). These doubts were motivated by the presence of the multiplicative anomaly in the zeta-regularized determinant ([3]). In this note, we shall investigate the question of the validity of (1) and, for this purpose, we first establish that

$$\det D=(-1)^{\nu_-}\det D_{(+)}\det D_{(-)},\quad D_{(\pm)}=\frac{|D|\pm D}{2}, \tag{2}$$

where D is a self-adjoint elliptic operator such that its spectral zeta-function

$$Z(D,s)=\operatorname{tr}\left(\frac{1}{2\pi i}\int_\gamma \lambda(\lambda I-D)^{-1}d\lambda\right),\quad \arg(-1)=i\pi, \tag{3}$$

admits an analytic continuation to $s=0$ which is holomorphic at $s=0$. $Z(D_{(\pm)},s)$ are similarly defined and ν_- in (2) stands for $Z(D_{(-)},0)$ (see [1]).

The determinant of D is defined by $\exp(-Z'(D,0))$, where $'$ denotes the derivative with respect to s. More precisely, (2) contains an ambiguity that comes from the many valuedness of the powers of -1. It is shown that the condition $Z(D_{(-)},0)$ being an integer is a necessary and sufficient condition for the uniqueness of the

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definition of $\det D$ (see Corollary 1.6 of Proposition 1.4). It is also shown that $\det D$ is uniquely defined if and only if it takes a real value, and $|\det D|$ is unique although $\det D$ is not unique. If the spectrum of D satisfies the symmetry condition, i.e., $\mathrm{Spec}\, D = \mathrm{Spec}\,(-D)$, where $\mathrm{Spec}\, D$ is the set of eigenvalues of D, then (1) reads

$$\det D = (-1)^{Z(D^2,0)/2} (\det D)^2. \tag{4}$$

If $m \neq 0, D + im$ is not self-adjoint, and we cannot use (1). But since $\lambda + im$ and $-\lambda + im$ are both eigenvalues of $D + im$, and $(\lambda + im)(-\lambda + im) = -(\lambda^2 + m^2)$, we define the sym-determinant sym-det$\,(D + im)$ by

$$\text{sym-det}\,(D + im) = (-1)^{Z(D^2+m^2,0)/2} \det(D^2 + m^2). \tag{5}$$

Then we have

$$\text{sym-det}\,(D + im) = \text{sym-det}\,(D - im), \tag{6}$$

$$(\text{sym-det}\,(D + im))(\text{sym-det}\,(D - im)) = (-1)^{Z(D^2+m^2,0)/2} \det(D^2 + m^2). \tag{7}$$

From (6) and (7), we can derive a condition for the validity of (1) as it is done in Sections 1 and 2. The sym-determinant may be different from the ordinary determinant. To discuss this problem, we compute $\det(D + im)$ as a function of m, provided D is a self-adjoint first-order operator on a compact manifold (cf. [1]). It is concluded that, due to the existence of residues of $Z(D, s)$, $\det(D + im)$ can be different from $\det(D - im)$ if $m \neq 0$. These are discussed in Sections 3 and 4.

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1. Zeta-determinants of self-adjoint operators

Let D be a non-degenerate elliptic pseudo-differential operator such that the spectrum of D does not meet the ray $R_\theta = \{re^{i\theta} | r \geq 0\}$. Then the complex power D_θ^s and the zeta-function $Z_\theta(D, s)$ of D are defined by

$$D_\theta^s = \frac{1}{2\pi i} \int_\gamma \lambda_\theta^s (\lambda I - D)^{-1} d\lambda, \tag{8}$$

$$Z_\theta(D, s) = \mathrm{tr}\, D_\theta^{-s}, \tag{9}$$

where $\lambda_\theta^s = |\lambda| \exp(is \arg \lambda)$, $\theta < \arg \lambda < \theta + 2\pi$. Here the contour γ is such that it encloses the spectrum of D (cf. [7, 8]). If D is self-adjoint and admits the spectral decomposition

$$Df = \sum \lambda_n (e_n, f) e_n, \tag{10}$$

we have

$$Z_\theta(D, s) = \sum_{\lambda_n > 0} \lambda_n^{-s} + \sum_{\lambda_n < 0} (-1)^{-s} |\lambda_n|^{-s}, \tag{11}$$

with $-1 = e^{\pi i}$, if $-\pi < \theta < 0$, and $-1 = e^{-\pi i}$, if $-2\pi < \theta < -\pi$.

If we fix $\arg 1 = 0$, then the only possible choices for $\arg(-1)$ are πi and $-\pi i$. In the remaining, we shall use the notation $Z(D,s)$ if $\arg(-1) = \pi i$, and $Z_{-}(D,s)$ if $\arg(-1) = -\pi i$. We introduce the following operators and zeta-functions related to D:

$$|D|f = \sum |\lambda_n|(e_n, f)e_n, \tag{12}$$

$$D(+) = (|D| + D)/2, \qquad D(-) = (|D| - D)/2, \tag{13}$$

$$Z(D(+), s) = \sum_{\lambda_n > 0} \lambda_n^{-s}, \qquad Z(D(-), s) = \sum_{\lambda_n < 0} |\lambda_n|^{-s}. \tag{14}$$

Notice that, by definition, $Z(|D|, s) = Z(D^2, s/2)$. We introduce the eta-function $H(D,s)$ of D, it is defined by $\sum_{\lambda_n} \operatorname{sgn}(\lambda_n)|\lambda_n|^{-s}$. Then we have that $Z(D(\pm), s) = (Z(|D|, s) \pm H(D,s))/2$. We assume that these zeta- and eta-functions are all continued analytically to 0 and are holomorphic at $s = 0$ (cf. [2, 5, 9]). We also denote the projections to the positive and negative eigenspaces of D by $P(+)$ and $P(-)$, respectively. If D is an n-th-order operator on a d-dimensional compact manifold, then the following facts are known ([5], for the case manifold with boundary, cf. [2, 9]).

Fact 1. $H(D,s)$ is continued meromophically to the whole complex plane, with possible poles at most of order 1 located at the points $s = d/n$, $(d-1)/n, \dots$

Fact 2. $H(D,s)$ is holomorphic at $s = 0$.

Fact 3. $H(D,s)$ takes real value if s is a real number. All the residues of $H(D,s)$ are real numbers.

Fact 1 is a classical result of Seeley and was proved by performing an asymptotic expansion of the heat kernel. Fact 2 is a famous result of Atiyah-Patodi-Singer and Gilkey. Fact 3 follows from Fact 1 and the following Lemma.

Lemma 1.1 *Let f be a meromophic function on the complex plane having poles on the real axis only and taking real values on some interval of the real axis. Then f is real valued on the real axis and the residues of f are real.*

Proof. We may assume that f takes real values on (a,b), where a and b are poles of f (or $a = -\infty$, $b = \infty$). Let the principal part of the Laurent expansion of f be $c_{-k}/(z-a)^k + \cdots$, then c_{-k} is given by $\lim_{x \to a+0}(x-a)^k f(x)$, so c_{-k} is a real number. Hence $f(z) - c_{-k}/(z-a)^k$ takes real values on (a,b). Repeating this, we conclude that $f(z) - \{\text{principal part of the Laurent expansion of } f \text{ at } a\} (= g(z))$ takes real values on (a,b). Then the coefficients of the Taylor expansion of $g(z)$ are all real numbers, so $g(z)$ takes real values on (a',b), $a' < a$ and f is holomorphic on (a',a). Hence $f(z)$ takes real values on $(a',a) \cup (a,b)$. By repeating this argument, the lemma follows.

From Fact 3, we have the following proposition.

Proposition 1.2 *If D is a positive elliptic operator on a compact manifold, then* $\det D$ *is a positive real number.*

Note. If we express $1 = e^{2m\pi i}$, where $m \neq 0$ is an integer, $Z(D,s)$ changes to $e^{-2ms\pi i}Z(D,s)$. Then, since $(e^{-2ms\pi i}Z(D,s))' = e^{-2ms\pi i}(-2m\pi i Z(D,s) + Z'(D,s))$, $\det D$ changes to $e^{2mZ(D,0)\pi i}\det D$. Proposition 1.2 can be formulated in a different way.

Proposition 1.3 *If D is a positive elliptic operator on a compact manifold, then* $\det D$ *does not depend on the expression of 1 (as a power of e), if and only if $Z(D,0)$ is an integer.*

From (11), we have

$$Z'(D,s) = \sum_{\lambda_n > 0} \left((-\log\lambda_n)\lambda_n^{-s} + e^{-s\pi i}(-\log\lambda_n - \pi i)|\lambda_n|^{-s} \right). \tag{15}$$

Hence using the above notations, we have

$$\begin{aligned} &\exp(-Z'(D,s)) = \\ &\quad \exp(iZ(D(-),0)\exp(-Z'(D(+),s))\exp(-Z'(D(-),s))\exp(e^{-s\pi i}). \end{aligned} \tag{16}$$

Proposition 1.4 *The zeta-determinant of a self-adjoint operator D is given by*

$$\det D = (-1)^{Z(D(-),0)}\det D(+)\det D(-). \tag{17}$$

Corollary 1.5 *Let* $\det D = \exp(-Z'(D,s))$. $\det D$ *is equal to* $\overline{\det D}$, *the complex conjugate of* $\det D$. *Consequently, we have* $|\det D| = \det|D|$ *(which is also equal to* $\det D(+)\det D(-)$*).*

Corollary 1.6 *The following statements are equivalent.*

i) $\det D$ *is a real number.*

ii) $\det D$ *does not depend on the expression of* -1.

iii) $Z(D(-),0)$ *is an integer.*

In general, $\det D$ depends on the expression of -1. The notations $\det D$ when $\arg(-1) = \pi i$ and $\det\_\, D$ when $\arg(-1) = -\pi i$ shall be used in the rest of the paper.

Note. As for $\det(-D)$, we have that $\det(-D) = (-1)^{H(D,0)}\det D$. In general, the right-hand side differs from $(-1)^{Z(|D|,0)}\det D$. But if $Z(D(-),0)$ is an integer, we have $(-1)^{H(D,0)} = (-1)^{Z(|D|,0)}$. That is, $\det(-D) = (-1)^{Z(|D|,0)}\det D$ if and only if $Z(D(-),0)$ is an integer. In Proposition 1.4, we assume $\arg\lambda_n = 0$ if $\lambda_n > 0$. If we release this assumption, we need to assume that $Z(D(+),0)$ is an integer to keep the uniqueness of $\det D$. Hence, together with Corollary 1.6, we need to assume that $H(D,0)$ and $Z(|D|,0)$ are integers, and mod 2 congruence of $H(D,0)$ and $Z(|D|,0)$.

2. Sym-determinant

In this section, we assume that the spectrum of D is symmetric, that is, if λ is an eigenvalue of D, then $-\lambda$ is also an eigenvalue (with same multiplicity). Then we have

$$Z(D_{(+)},s) = Z(D_{(-)},s) = Z(D,s/2)/2. \tag{18}$$

By combining (18) and Proposition 1.4, we obtain a simple consequence.

Proposition 2.1 *If the spectrum of D is symmetric, then*

$$\det D = (-1)^{Z(D_{(-)},0)}(\det D_{(+)})^2, \tag{19}$$

and $\det D$ *is equal to* $\det\_ D$ *if and only if* $Z(D,0)$ *is an even integer.*

Corollary 2.2 $(\det D)^2 = \det(D^2)$ *if and only if* $Z(D^2,0)$ *is an even integer. More precisely, we have*

$$\begin{aligned} \det D &= \sqrt{\det(D^2)}, & &\text{if } Z(D,0) \equiv 0 \mod 4, \\ \det D &= -\sqrt{\det(D^2)}, & &\text{if } Z(D,0) \equiv 2 \mod 4. \end{aligned} \tag{20}$$

We shall apply this result to the determinant of $D+im$. Usually, $\det(D+im)$ is defined by using the symmetry property of the spectrum and involves the following change of the order of multiplication (cf. [3])

$$(\lambda_n + im)(-\lambda_n + im) = (-\lambda_n - im)(\lambda_n - im)) = -(\lambda_n^2 + m^2). \tag{21}$$

If we accept this reordering of factors, we must have $\det(D+im)$=$\det(D-im)$. But since we are working with regularized infinite product, the change of the order of multiplication must have some effect on the answer. So we shall use the following notation.

Definition 2.3 *We define* sym-det $(D+im)$ *by*

$$\text{sym-det}\,(D+im) = (-1)^{Z(D_{(+)}^2+m^2,0)}\det(D_{(+)}^2+m^2). \tag{22}$$

This definition is motivated by the fact that the rearrangement (21) regards "det"$(D+im)$ to be $\det\big(-(D_{(+)}^2+m^2)P_{(+)}\big)$. In the rest, we use the notation $D_{(+)}^2+m^2$, instead of $(D_{(+)}^2+m^2)P_{(+)}$ for the sake of simplicity. Since $\det(D_{(+)}^2+m^2) = \det(D_{(-)}^2+m^2)$ and $\det(D_{(+)}^2) = (\det D_{(+)})^2$, we have

$$\text{sym-det}\,(\not{D} + im) = \text{sym-det}\,(\not{D} - im), \tag{23}$$

$$\text{sym-det}\,\not{D} = \det \not{D}\,. \tag{24}$$

By definition, $Z(\not{D}^2+m^2)$ is equal to $2Z(\not{D}_{(+)}^2+m^2)$ $(=2Z(\not{D}_{(-)}^2+m^2))$. Hence we have

$$\det(\not{D}^2+m^2) = (\det(\not{D}_{(+)}^2+m^2))^2. \tag{25}$$

By (25) and the equality $Z(\not{D}_{(+)}^2+m^2,0) = Z(\not{D}^2+m^2,0)/2$, we obtain

Theorem 2.4 *The following equalities are holding.*

$$\text{sym-det}\,(\not{D}+im)=\sqrt{\det(\not{D}^2+m)},\qquad \text{iff } Z(\not{D}^2+m^2,0)\equiv 0 \mod 4, \tag{26}$$

$$\text{sym-det}\,(\not{D}+im)=-\sqrt{\det(\not{D}^2+m^2)},\quad \text{iff } Z(\not{D}^2+m^2,0)\equiv 2 \mod 4. \tag{27}$$

We also have

$$|\text{sym-det}\,(D+im)|=|\sqrt{\det(D^2+m^2)}|, \tag{28}$$

if D is defined on a compact manifold.

Note. The sym-determinant is also defined for $D+m$ and the computation of sym-det $(D+m)=\det(-D_{(+)}{}^2-m^2)$ is not simple, as $-(D_{(+)}{}^2-m^2)$ is not a positive operator. As for the relation between $\det(-(D_{(+)}{}^2-m^2))$ and $\det(-(D^2-m^2))$, set $c=\#\{\lambda|\lambda$ is an eigenvalue value of $D_{(+)}$ such that $\lambda<m\}$, we have $\det(-(D_{(+)}{}^2-m^2))=(-1)^c\{\det(-(D_{(+)}{}^2-m^2))\}$. So we need to take into account the contribution from c. Theorem 2.4 and (23) give an answer to the queries stated in the Introduction. But there still remains the problem to know whether sym-det$(\not{D}+im)=\det(\not{D}+im)$ or not if $m\neq 0$. To give an answer to this problem, we calculate the variation of $\det(D+m)$ with respect to m.

Note. As for the operator $iD+m$, sym-det $(iD+m)$ is defined to be $\det(D_{(+)}{}^2+m^2)$, which is equal to $\sqrt{\det(D^2+m^2)}$ if we fix $\arg\lambda_n=0$ (cf. [10]). But if the choice $\arg\lambda_n=2m\pi i$, $m\neq 0$, is allowed, $Z(D^2+m^2,0)/2$ is still an integer.

3. Variation of $Z(D+m,0)$ with respect to m

We assume that D is a first-order operator defined on a d-dimensional compact manifold. Then $H(D,s)$ and $Z(|D|,s)=H(D,s/2)$ are continued meromorphically to the whole complex plane with possible poles at $s=d,d-1,\dots,$ of order 1, and are holomorphic at $s=0$ ([5], for the operators on manifold with boundary, cf. [2, 9]). In the remaining, we shall use only these properties of $H(D,s)$ and $Z(|D|,s)$. Since $Z(D,s)_{(\pm)}=(Z(|D|,s)\pm H(D,s))/2$, these functions have the same properties as $H(D,s)$ and $Z(|D|,s)$. Moreover we have

$$\operatorname{Res}_{s=m} Z(|D|,s)=\operatorname{Res}_{s=m} Z(D_{(+)},s)+\operatorname{Res}_{s=m} Z(D_{(-)},s), \tag{29}$$

$$\operatorname{Res}_{s=m} H(D,s)=\operatorname{Res}_{s=m} Z(D_{(+)},s)-\operatorname{Res}_{s=m} Z(D_{(-)},s). \tag{30}$$

For a positive number N, we set

$$Z(D_{(+)},N,s)=\sum_{\lambda_n>N}\lambda_n^{-s}=Z(D_{(+)},s)-\sum_{0<\lambda_n\leq N}\lambda_n^{-s}, \tag{31}$$

$$Z(D_{(-)},N,s)=\sum_{\lambda_n<-N}|\lambda_n|^{-s}=Z(D_{(-)},s)-\sum_{0>\lambda_n\geq -N}|\lambda_n|^{-s}. \tag{32}$$

If $|m|<|\lambda_1|$, where λ_1 the smallest eigenvalue of $|D|$, we have

$$Z(D_{(+)}+m,s)=\sum(\lambda_n+m)^{-s}=\sum\lambda_n^{-s}(1+m/\lambda_n)^{-s}. \tag{33}$$

Since $(1+m/\lambda_n)^{-s} = 1 - s(m/\lambda_n) + \cdots + (-1)^k\bigl(s\cdots(s+k-1)/k!\bigr)(m/\lambda_n)^k + \cdots$, we have by (33)

$$\begin{aligned} Z(D(+)+m,s) =& Z(D(+),s) - sZ(D(+),s+1)m + \cdots \\ & \cdots + (-1)^k(s\cdots(s+k-1))/k!\, Z(D(+),s+k)m^k + \cdots . \end{aligned} \tag{34}$$

By (34), we obtain for $|m| < |\lambda_1|$,

$$\begin{aligned} Z(D(+)+m,0) =& Z(D(+),0) - \operatorname{Res}_{s=1} Z(D(+),s)m + \cdots \\ & \cdots + (-1)^k \operatorname{Res}_{s=k} Z(D(+),s)(m^k/k) + \cdots \\ & \cdots + (-1)^d \operatorname{Res}_{s=d} Z(D(+),s)(m^d/d). \end{aligned} \tag{35}$$

A similar formula holds for $D(-)$,

$$\begin{aligned} Z(D(-)+m,0) =& Z(D(-),0) + \operatorname{Res}_{s=1} Z(D(-),s)m + \cdots \\ & \cdots + \operatorname{Res}_{s=k} Z(D(-),s)(m^k/k) + \cdots \\ & \cdots + \operatorname{Res}_{s=d} Z(D(-),s)(m^d/d). \end{aligned} \tag{36}$$

If m is real and $|m| < |\lambda_1|$, we have

$$\begin{aligned} Z(|D+m|,s) &= Z(D(+)+m,s) + Z(D(-)-m,s), \\ H(D+m,s) &= Z(D(+)+m,s) - Z(D(-)-m,s). \end{aligned} \tag{37}$$

Hence we obtain

$$\begin{aligned} Z(|D+m|,0) =& Z(|D|,0) + \operatorname{Res}_{s=1} H(D,s)m \\ & + \operatorname{Res}_{s=2} Z(|D|,s)(m/2) + \cdots + (-1)a(d)(m^d/d), \end{aligned} \tag{38}$$

$$a(d) = \begin{cases} \operatorname{Res}_{s=d} Z(|D|,s), & \text{if } d \text{ is even}, \\ \operatorname{Res}_{s=d} H(D,s), & \text{if } d \text{ is odd}. \end{cases} \tag{39}$$

$$\begin{aligned} H(D+m,0) =& H(D,0) + \operatorname{Res}_{s=1} Z(|D|,s)m \\ & + \operatorname{Res}_{s=2} H(D,s)(m/2) + \cdots + (-1)b(d)(m^d/d), \end{aligned} \tag{40}$$

$$b(d) = \begin{cases} \operatorname{Res}_{s=d} H(D,s), & \text{if } d \text{ is even}, \\ \operatorname{Res}_{s=d} Z(|D|,s), & \text{if } d \text{ is odd}. \end{cases} \tag{41}$$

Equations (38) and (40) are proved by assuming that m is real, but their right-hand sides are polynomials in m, hence holomorphic in m. On the other hand, taking the contours in the definitions of the complex powers of $|D+m|$, etc., to be $\{x+ai \mid -\infty < x < \infty\} \cup \{x-ai \mid \infty > x > -\infty\}$, the right-hand sides of (38) and (40) are holomorphic in m if $|\operatorname{Im} m| < a$. Therefore (38) and (40) are valid though m is a complex number provided $|m| < |\lambda_1|$. Before computing $Z(|D+m|,0)$, etc., for large $|m|$, let us note that for $j = 1, \ldots, d$,

$$\begin{aligned} \operatorname{Res}_{s=j} Z(D(+),N,s) &= \operatorname{Res}_{s=j} Z(D(+),s), \\ \operatorname{Res}_{s=j} Z(D(-),N,s) &= \operatorname{Res}_{s=j} Z(D(-),s). \end{aligned} \tag{42}$$

Because $\sum_{0<\lambda_n\leq N}\lambda_n^{-s}$, etc., are entire functions of s. By (35) and (42), to set $Z(D(+)+m,\text{“}N\text{”},s)=\sum_{\lambda_n>N}(\lambda_n+m)$, etc., we have for $|m|<N$,

$$\begin{aligned}Z(D(+)+m,\text{“}N\text{”},0) =&Z(D(+),N,0)-\operatorname{Res}_{s=1}Z(D(+),s)m+\cdots\\ &\cdots+(-1)^d\operatorname{Res}_{s=d}Z(D(+),s)(m^d/d).\end{aligned} \tag{43}$$

On the other hand, by the definition of $Z(D(+)+m,\text{“}N\text{”},s)$, etc., we have

$$Z(D(+)+m,\text{“}N\text{”},0)=Z(D(+)+m,0)-\#\{\lambda_n|0<\lambda_n<N\}, \tag{44}$$

$$Z(D(+),0)=Z(D(+),0)-\#\{\lambda_1|0<\lambda_n<N\}. \tag{45}$$

Hence (35) is valid for any m. As for (38) and (40), answers may be different if $|m|$ is large. But the differences must be even numbers. So we have ([1])

$$\begin{aligned}Z(|D+m|,0)\equiv&Z(|D|,0)-\operatorname{Res}_{s=1}H(D,s)m+\cdots+(-1)^d a(d)(m^d/d)\quad \bmod 2,\\ H(D+m,0)\equiv&H(D,0)-\operatorname{Res}_{s=1}Z(|D|,s)m+\cdots+(-1)^d b(d)(m^d/d)\quad \bmod 2.\end{aligned}$$

By the same calculation, we get the following formula, which is valid for any m, and useful to the application of Theorem 2.4.

$$\begin{aligned}Z(\not{D}^2+m^2,0)=&Z(\not{D}^2,0)-\operatorname{Res}_{s=1}Z(\not{D}^2,s)m^2+\cdots\\ &\cdots+(-1)^{2[d/2]}\operatorname{Res}_{s=2[d/2]}Z(D,s)(m^{2[d/2]}[d/2]).\end{aligned} \tag{46}$$

Here $[d/2]$ stands for Gauss' symbol. Although $Z(\not{D}^2,s)$ has poles at $s=d/2$, $(d-1)/2,\ldots$, (46) shows that residues at $s=j/2,j,\ldots$ are odd numbers and they do not contribute to the value of $Z(\not{D}^2+m^2,0)$ (cf. [3, 4]). The values of m for which the second equality of (1) is determined by $\not{D}$. If $d=1$, $Z(\not{D}^2+m^2)$ is constant in m, so the validity of (1) is determined by $\not{D}$. If $d=2$ or 3, $Z(\not{D}^2+m^2)$ is linear in m. If $d=2k$, $\operatorname{Res}_{s=k}Z(\not{D}^2,s)>0$, so the coefficient of m is negative if $d=2$. If $d=4$ or 5, $Z(\not{D}^2,0)$ is a quadratic form in m. The coefficient of m^4 is positive if $d=4$. Since the equation $Z(\not{D}^2+m^2,0)=4n$ (or $4n+2$), n is an integer, should have real positive solutions, these remarks have meanings.

4. Variation of det$(D+m)$ with respect to m

In this section, we shall compute $\det(D+m)$. Since the orders of the poles of $Z(D,s)$ etc., are at most 1, we can make the expansion:

$$Z(D(+),s+j)=a(j,+)_{-1}s^{-1}+a(j,+)_0+a(j,+)_1s+\cdots. \tag{47}$$

By (47), we have

$$\frac{d}{ds}(sZ(D(+),s+1))|_{s=0}=a(1,+)_0. \tag{48}$$

$$\begin{aligned}&\frac{d}{ds}(s(s+1)\cdots(s+j-1)/j!\,Z(D(+),s+j))|_{s=0}\\ &\quad=\frac{1}{j}(1+\cdots+1/(j-1))a(j,+)_{-1}+(1/j)a(j,+)_0,\quad j>2.\end{aligned} \tag{49}$$

By (35), (49), and (49), we obtain

$$
\begin{aligned}
Z'(D(+)+m,0) =& Z'(D(+),0) + \sum_{j=2}^{d}(1/j!)(1+\cdots+1/(j-1))a(j,+)_{-1}m^j \\
&+ \sum_{j=1}^{\infty}(-1)^j a(j,+)/jm^j.
\end{aligned}
\tag{50}
$$

Since $Z(D(+),N,s+j) = Z(D(+),s+j) - \sum_{0<\lambda_n\leq N}\lambda_n^{-(s+j)}$, we have

$$
\begin{aligned}
Z'(D(+)+m,\cdots,0) =& Z'(D(+),\text{``}N\text{''},0) + A(+,m) \\
&+ \sum_{j=1}^{\infty}(-1)^j(1/j)(a(j,+)_0 - \sum_{0<\lambda_n\leq N}\lambda_n^{-j})m^j,
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
Z'(D(-)+m,\cdots,0) =& Z'(D(-),\text{``}N\text{''},0) + A(-,m) \\
&+ \sum_{j=1}^{\infty}(-1)^j(1/j)(a(j,-)_0 - \sum_{0>\lambda_n\geq N}|\lambda_n|^{-j})m^j.
\end{aligned}
\tag{52}
$$

Here $A(+,m)$ and $A(-,m)$ are given by

$$
A(\pm,m) = \sum_{j=2}^{d}(1/j)(1+\cdots+1/(j-1))a(j,\pm)_{-1}m^j.
\tag{53}
$$

We note that $a(1,+)_{-1}$ do not affect to the values of $A(+,m)$ and $Z'(D(+)+m,0)$. We also note that since $|a(j,+)_0 - \sum_{0<\lambda_n\leq N}\lambda_n^{-j}| = O(N^j)$ if $j>d$, the infinite sums in (51) and (52) converge if $|m|<N$. On the other hand, we have

$$
Z'(D(+),N,0) = Z'(D(+),0) - \sum_{0<\lambda_n\leq N}\log\lambda_n,
\tag{54}
$$

$$
Z'(D(+)+m,\text{``}N\text{''},0) = Z'(D(+)+m,0) - \sum_{0<\lambda_n\leq N}\log(\lambda_n+m).
\tag{55}
$$

Hence we obtain for $|m|<N$,

$$
\begin{aligned}
Z'(D(+)+m,0) =& Z'(D(+),0) + \sum_{0<\lambda_n\leq N}\log(1+(m/\lambda_n)) + \\
&+ \sum_{j=1}^{\infty}(-1)^j(1/j)(a(j,+)_0 - \sum_{0<\lambda_n\leq N}\lambda_n^{-j})m^j + A(+,m),
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
Z'(D(-)-m,0) =& Z'(D(-),0) + \sum_{0>\lambda_n\geq -N}\log(1-(m/|\lambda_n|)) + \\
&+ \sum_{j=1}^{\infty}(1/j)(a(j,-)_0 - \sum_{0>\lambda_n\geq N}|\lambda_n|^{-j})m^j + A(-,m).
\end{aligned}
\tag{57}
$$

By (17), $\det(D+m) = (-1)^{(Z(|D-m|,0)-H(D-m,0))/2}\det(D(+)+m)\det(D(-)-m)$. Hence by (56) and (57), we have

$$\begin{aligned}
&\det(D+m) = \\
&(-1)^{(Z(|D-m|,0)-H(D-m,0))/2}\det D \prod_{|\lambda_n|\leq N}(1+(m/\lambda_n))\exp(A(+,m)+A(-,-m)) \\
&\exp\Big(\sum_{j=1}^{d}((-1)^j/j)\big(a(j,+)_0+(-1)a(j,-)_0-\sum_{|\lambda_n|\leq N}|\lambda_n|^{-j}\big)m^j\Big). \qquad (58)
\end{aligned}$$

Here we have

$$\begin{aligned}
&(Z(|D-m|,0)-H(D-m,0))/2 \\
&= (Z(|D|,0)-H(D,0))/2-\big((\mathrm{Res}_{s=1}Z(|D|,s)-\mathrm{Res}_{s=1}H(D,s))/2\big)m+\cdots \\
&\qquad \cdots+(-1)^d\big(\mathrm{Res}_{s=d}Z(|D|,s)-\mathrm{Res}_{s=d}H(D,s)\big)/2(m^d/d) \\
&= Z(D(-),0)\cdots\mathrm{Res}_{s=1}Z(D(-),s)m+\cdots+(-1)^d\mathrm{Res}_{s=d}Z(D(-),s)(m^d/d),
\end{aligned}$$

$$\begin{aligned}
&A(+,m)+A(-,-m) \\
&= \sum_{j=2}^{d}(1/j)(1+\cdots+1/(j-1))(a(j,+)_{-1}+(-1)a(j,-)_{-1})m \\
&= (1/2)\mathrm{Res}_{s=2}Z(|D|,s)m^2+\cdots+(-1)^d(1/d)(1+\cdots+1/(d-1))b(d)m^d.
\end{aligned}$$

Equation (58) is valid only for $|m|<N$. But since

$$\begin{aligned}
\sum_{j=1}^{\infty}(-1)^{-1}(1/j)\Big(\sum_{\lambda_n\leq N}|\lambda_n|^{-j}\Big)m^j &= \sum_{\lambda_n\leq N}\Big(\sum_{j=1}^{\infty}(-1)^j(1/j)(m/|\lambda_n|)^j \\
&= -\sum_{\lambda_n\leq N}\log(1+m/|\lambda_n|),
\end{aligned}$$

if $|m|<\min\{\lambda_n|\lambda_n>0\}$, we have on the same domain

$$\begin{aligned}
Z'(D(+)+m,0) =&Z'(D(+),0)-\sum_{\lambda_n\leq N}\log(1+m/|\lambda_n|) \\
&\qquad +A(+,m)+\sum_{j=1}^{\infty}(-1)^j(1/j)a(j,+)_0m^j+\log(1+m/|\lambda_n|) \\
=&Z'(D(+),0)+A(+,m)+\sum_{j=1}^{\infty}(-1)^j(1/j)a(j,+)_jm^j.
\end{aligned}$$

Hence $\det(D(+)+m)$ is continued holomorphically to the whole complex plane. Similarly, $\det(D(-)+m)$ is continued holomorphically to the whole complex plane. Therefore the function

$$\text{“det”}(D+m) = (-1)^{\{Z(|D-m|,0)-H(D-m,0)\}/2}\det(D(+)+m)\det(D(-)-m) \qquad (59)$$

is an entire function of m. "det"$(D+m)$ may differ by a sign from $\det(D+m)$ if $|m|$ is large, because the difference between $\det(D+m)$ and "det"$(D+m)$ comes from the relative sign between $\lambda_n - m$ and $m-\lambda_n$. So we may use "det"$(D+m)$ instead of $\det(D+m)$ ([1]). As a function of m, (59) is written

$$\begin{aligned}\text{“det”}(D+m) &= (-1)^{Z(D(-)-m,0)} \prod_{|\lambda_n|\leq N} (1+m/\lambda_n) \exp(A(+,m)+A(-,m)) \times \\ &\times \exp(\sum_{j=1}^{\infty} (-1)^j (1/j)(a(j,+)_0 + (-1)^j a(j,-)_0) - \sum_{|\lambda_n|\leq N} |\lambda_n|^{-j}) m^j.\end{aligned} \tag{60}$$

If the spectrum of D is symmetric, we have $\prod_{|\lambda_n|\leq N}(1+m/\lambda_n) = (1-m^2/\lambda_n^2)$ and $a(j,+)_0 = a(j,-)_0$. Hence we have

$$\begin{aligned} & Z(D(-)-m,0) \\ &= Z(D(-),0) - \operatorname{Res}_{s=1} Z(D(-),s)m + \cdots + (-1)^d \operatorname{Res}_{s=d} Z(D(-),s)(m^d/d) \\ &= \frac{1}{2}(Z(|D|,0) - \operatorname{Res}_{s=1} Z(|D|,s)m + \cdots + (-1)^d \operatorname{Res}_{s=d} Z(|D|,s)(m^d/d)),\end{aligned} \tag{61}$$

$$A(+,m)+A(-,m) = \sum_{j=2}^{d} \operatorname{Res}_{s=j} Z(|D|,s)(1/2j)(1+\cdots+1/(2j-1))m^j. \tag{62}$$

Since $Z(|D|,s) = Z(D^2,s/2)$, we have $\operatorname{Res}_{s=j}(|D|,s) = 2\operatorname{Res}_{s=j/2}(D^2,s)$, so Equations (61) and (62) become

$$\begin{aligned} Z(D(-)-m,0) = Z(D^2,0)/2 - \operatorname{Res}_{s=1/2} Z(D^2,s)m \\ \cdots + (-1)^d \operatorname{Res}_{s=d/2} Z(D^2,s)(m^2/d),\end{aligned}$$

$$A(+,m)+A(-,m) = \sum_{j=2}^{d} \operatorname{Res}_{s=d/2} Z(D^2,s)(1/j)(1+\cdots+1/(2j-1))m^j.$$

Hence (60) becomes

$$\begin{aligned} &\text{“det”}(D+m) = \\ &\exp\Big((Z(D^2,0)/2 - \operatorname{Res}Z(D^2,s)m + \cdots + (-1)^d \operatorname{Res}Z(D^2,s)(m^d/d))\pi i\Big) \times \\ &\times \prod_{|\lambda_n|\leq N} (1-m^2/\lambda_n^2) \exp\Big(\sum_{j=2}^{d} \operatorname{Res}_{s=d/2} Z(D^2,s)(1/j)(1+\cdots+1/(2j-1))m^j\Big) \times \\ &\times \exp\Big(\sum_{j=1}^{\infty}((-1)^j/j)(1+(-1)^j)a(j,+)_0 - 2\sum_{0<\lambda_n\leq N} \lambda_n^{-j})m^j\Big).\end{aligned}$$

Since this formula gives $\det(D+m)$ if $|m|$ is small, $\det(D+im)$ may different from $\det(D-im)$, by virtue of the presence of residues of $Z(D,s)$. It also shows that sym-det$(D+im)$ may be different from $\det(D+im)$ if $m \neq 0$.

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Projectable star-products

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Abstract. We give a contravariant version of Fedosov's construction of star-products on symplectic manifolds. This result is used to provide an alternative construction of star-products on Abelian Poisson Lie groups.

Keywords: star-product, deformation quantization, Fedosov quantization

Mathematics Subject Classifications (2000): 58F06, 81S99, 81S10

To the memory of Moshé Flato

1. Introduction

The theory of formal deformation quantization of Poisson manifolds was introduced by Bayen et al. [1]. The main notion of this theory is the concept of a *star-product* on a Poisson manifold N, that is, a particular associative algebraic structure $*$ on the space $C^\infty(N)[[h]]$ of formal power series in h whose coefficients are smooth functions on N. The general question of the existence of such a product for *symplectic manifolds* has been completely solved by several authors, using various techniques [3, 5, 10]. Recently, M. Kontsevich has proved the existence of star-products on arbitrary finite-dimensional Poisson manifolds [8].

In this note, we consider whether a star-product can be constructed on an arbitrary Poisson manifold N by "*projecting*" a star-product from a symplectic manifold M. More precisely, let $*_M$ be a star-product on M, and let $\phi : M \to N$ be a surjective smooth submersion. A star-product $*_M$ on M is said to be projectable by ϕ if $\phi^*C^\infty(N)[[h]]$ is a subalgebra of $(C^\infty(M)[[h]], *_M)$; such star-product induces a star-product in N.

In Fedosov's geometrical construction of star-products on a symplectic manifold M one builds a *formal algebra bundle*, the so-called *Weyl bundle*, and a flat connection on it. One then shows that the algebra of flat sections of this bundle is in bijection with $C^\infty(M)[[h]]$. Therefore, one may attempt to choose the connection in such a way that a natural subalgebra of the algebra of flat sections is identified with $\phi^*C^\infty(N)[[h]]$. Here we shall show that this can be done in the particular case of *Abelian Poisson Lie groups*. In order to do this, we use a "contravariant" version of Fedosov's construction which seems to have some advantages if one wishes to project star-products.

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2. Preliminaries

Let (M,π) be a *Poisson manifold*, let $\Omega_p(M)$ and $\Omega^p(M)$ be the spaces of p-vectors and p-forms on M. Let $\pi^\# : T^*M \to TM$ be the bundle map defined by $\langle \pi^\#(\alpha),\beta\rangle = \pi(\alpha,\beta)$, $\alpha,\beta \in \Omega^1(M)$. For simplicity we write $\pi^\#(\alpha) = \alpha^\#$. A Poisson manifold is called symplectic if the map $\pi^\#$ is an isomorphism, in such a case, we shall denote the inverse map by $\flat$. The existence of a Lie bracket of 1-forms on (M,π) allows to define a coboundary operator $\tilde{\partial}$ (i.e., $\tilde{\partial}^2 = 0$) on $\Omega_*(M) := \oplus_{p=0}\Omega_p(M)$ by setting

$$\begin{aligned}\tilde{\partial}P(\alpha_o,\dots,\alpha_p) = &\sum_{i=0}^{p}(-1)^i\alpha_i^\#(P(\alpha_0,\dots,\hat{\alpha}_i,\dots,\alpha_p))\\ &+\sum_{i<j}(-1)^{i+j}P(\{\alpha_i,\alpha_j\},\alpha_0,\dots,\hat{\alpha}_i,\dots,\hat{\alpha}_j,\dots,\alpha_p).\end{aligned}$$

Definition 1. **([11])** *Let $p : E \to M$ be a vector bundle over (M,π). A* contravariant connection *is a map that assigns to each $\alpha \in \Omega^1(M)$ and $s \in \Gamma(E)$ another smooth section $\nabla_\alpha s \in \Gamma(E)$ such that, for $f \in C^\infty(M)$,*

$$\nabla_\alpha f = \alpha^\# f, \quad \nabla_{f\alpha}s = f\nabla_\alpha s, \quad \textit{and} \quad \nabla_\alpha(fs) = f\nabla_\alpha s + (\alpha^\# f)s.$$

We use the contravariant connection to define a *contravariant differential* of E-valued vectors, i.e., a map $\tilde{\partial}^\nabla : \Gamma(\Omega_p(M)\otimes E) \to \Gamma(\Omega_{p+1}(M)\otimes E)$, $p \geq 0$, by means of the formula

$$\begin{aligned}\tilde{\partial}^\nabla s(\alpha_o,\dots,\alpha_p) = &\sum_{i=0}^{p}(-1)^i\nabla_{\alpha_i}(s(\alpha_0,\dots,\hat{\alpha}_i,\dots,\alpha_p))\\ &+\sum_{i<j}^{p}(-1)^{i+j}s(\{\alpha_i,\alpha_j\},\alpha_0,\dots,\hat{\alpha}_i,\dots,\hat{\alpha}_j,\dots,\alpha_p).\end{aligned}$$

3. Contravariant construction of star-products

Definition 2. **([1])** *A $*$-product on (M,π) is a bilinear map $C^\infty(M)\times C^\infty(M) \to C^\infty(M)[[h]]$ defined by*

$$(f,g) \mapsto f*g = \sum_{n=0}^{\infty} h^n C_n(f,g),$$

where $C_0(f,g) = fg$, $C_1(f,g) = \pi(df,dg)$, and for $n \geq 2$, C_n is a bilinear map with values in $C^\infty(M)$, which vanishes on the constants and satisfy the following relation

$$\sum_{r+s=k} C_r(C_s(f,g),h) = \sum_{r+s=k} C_r(f,C_s(g,h)), \quad k \geq 0.$$

Let π be the Poisson bivector on $\mathbb{R}^{2n}$ associated to the *canonical symplectic form* ω. Let us consider the coordinates $(y^1,\dots,y^{2n})$ of $\mathbb{R}^{2n}$ as elements of $\mathbb{R}^{2n^*}$, let h be a formal parameter, and let $\pi^{ij}=\pi(y^i,y^j)$. We assign the degree 1 to each y^i and degree 2 to the parameter h.

Definition 3 *The formal Weyl algebra W associated to $(\mathbb{R}^{2n},\pi)$ is the space of formal series*

$$W=\Big\{a(y,h)=\sum_{m=0}^{\infty}\sum_{2k+p=m}a_{k,i_1,\dots,i_p}h^k y^{i_1}\cdots y^{i_p}\,\Big|$$

$$a_{k,i_1,\dots,i_p}\in\mathbb{R}\text{ is symmetric in }i_1,\dots,i_p\Big\}$$

with the associative product (the Weyl product)

$$a\circ b=\sum_{k=0}^{\infty}\left(\frac{h}{2}\right)^k\frac{1}{k!}\pi^{i_1j_1}\cdots\pi^{i_kj_k}\frac{\partial^k a}{\partial y^{i_1}\cdots\partial y^{i_k}}\frac{\partial^k b}{\partial y^{j_1}\cdots\partial y^{j_k}}.$$

The contragredient action of the natural action of the *symplectic group* $Sp(2n,\mathbb{R})$ on $\mathbb{R}^{2n}$ extends naturally to an action ρ on W, which respects the Weyl product, that is to say, $Sp(2n,\mathbb{R})$ acts by automorphisms of W.

Definition 4 *Let (M,π) be a symplectic manifold, and let $F(M)$ be its symplectic frame bundle. The associated bundle $\mathscr{W}:=F(M)\times_{Sp(n,\mathbb{R}),\rho}W$ is called the (formal)* Weyl bundle.

The space of sections $\Gamma(\mathscr{W})$ with respect to pointwise multiplication forms an associative algebra. Furthermore, one defines the product of $X\otimes a$, $X'\otimes a'\in\Gamma(\Omega_*(M)\otimes\mathscr{W})$ by setting $(X\otimes a)\circ(X'\otimes a')=X\wedge X'\otimes a\circ a'$, and a graded Lie algebra structure on $\Gamma(\Omega_*(M)\otimes\mathscr{W})$ by setting

$$[s_1,s_2]=s_1\circ s_2-(-1)^{q_1q_2}s_2\circ s_1,\qquad s_i\in\Gamma(\Omega_{q_i}(M)\otimes\mathscr{W}).$$

Following [5], we define two linear operators $\delta:\Gamma(\Omega_q(M)\otimes\mathscr{W})\to\Gamma(\Omega_{q+1}(M)\otimes\mathscr{W})$ and $\delta^*:\Gamma(\Omega_q(M)\otimes\mathscr{W})\to\Gamma(\Omega_{q-1}(M)\otimes\mathscr{W})$, $q\geq 1$, as follows. Let $\hat{\pi}:=\sum_{i=1}^{2n}\frac{\partial}{\partial x^i}\otimes y^i$

$$\delta a=\frac{1}{h}[\hat{\pi},a],\qquad \delta^*(a)=\sum_{k=1}^{2n}y^k i_{(\frac{\partial}{\partial x^k})^\flat}a,\quad a\in\Gamma(\Omega_q(M)\otimes\mathscr{W}),\tag{1}$$

where i stands for the interior product of a 1-form and a p-vector. If $q=0$, we set $\delta^*a=0$.

Lemma 1 *The operators δ and δ^* do not depend on the choice of local coordinates, and satisfy $\delta^2={\delta^*}^2=0$. Furthermore, the operator δ is a graded derivation.*

Note that any $a \in \Gamma(\Omega_*(M) \otimes \mathscr{W})$ can be written (locally) as

$$a = \sum_{2k+p \geq 0, q \geq 0} h^k a_{k,j_1,\dots,j_q,i_1,\dots,i_p} \frac{\partial}{\partial x^{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_q}} \otimes y^{i_1} \cdots y^{i_p} =: \sum_{q \geq 0, p \geq 0} a_{qp}, \quad (2)$$

where the coefficients are mixed tensors symmetric in $i_1, \dots, i_p$ and antisymmetric in $j_1, \dots, j_q$. In particular, $a_{00} = \sum_{k \geq 0} h^k a_k$, and so if we define the operator δ^{-1} acting on a monomial a_{qp} by $\delta^{-1}(a_{00}) = 0$, and $\delta^{-1}(a_{qp}) = \frac{1}{q+p}\delta^*(a)$ if $q+p > 0$, then we get the decomposition

$$(\delta\delta^{-1} + \delta^{-1}\delta)(a) = a - a_{00}, \quad a \in \Gamma(\Omega_*(M) \otimes \mathscr{W}). \quad (3)$$

Flat connections on the Weyl bundle

An *almost symplectic connection* ∇' on a symplectic manifold (M, π) induces a contravariant connection on TM by setting $\nabla_\alpha X := \nabla'_{\alpha^\#} X$. Moreover, using the contravariant differentiation of the coefficients in (2), we define a contravariant differential on $\mathscr{W}$ as an operator $\partial : \Gamma(\mathscr{W}) \to \Gamma(\Omega_1(M) \otimes \mathscr{W})$ by setting $\partial a = \sum_i \frac{\partial}{\partial x^i} \otimes \nabla_{dx^i} a$. A calculation yields that in a *Darboux coordinate system*, ∂ is given by

$$\partial a = \tilde{\partial} a + \frac{1}{2h}\Big[\sum_{i,j,r=1}^{2n} \pi^{ir} \Gamma^r_j \otimes y^i y^j, a\Big], \quad a \in \Gamma(\mathscr{W}), \quad (4)$$

where Γ^r_j is the connection matrix of 1-vectors (of the connection ∇). A simple calculation shows that the curvature of ∂ is given by $\partial^2 a = \frac{1}{h}[\bar{R}, a]$, where $\bar{R} = \frac{1}{2}\sum_{r,l,k=1}^{2n} \pi^{rl} R^l_k \otimes y^r y^k$ and R^i_j is the curvature matrix of 2-vectors.

Now we look for a flat connection D of the type $D = \partial - \delta + \frac{1}{h}[r, \cdot]$, where r is a $\mathscr{W}$-valued 1-vector. The condition for D to be flat is that the element $\pi + \partial\hat{\pi} + \bar{R} + \partial r - \delta r + \frac{1}{2h}[r, r]$ has to be a central element of $\Gamma(\Omega_* \otimes \mathscr{W})$.

Note that there is a filtration in the Weyl algebra $W \supset W_1 \supset W_2 \supset \cdots$ with respect to the total degree $2k + p$ of the terms of the series (2): $W_k =$\{elements of degree $\geq k$\}. The proof of the following lemma follows exactly the same pattern as the one given in [5].

Lemma 2 *If $r_0 \in \Gamma(\Omega_1(M) \otimes W_k)$, $k \geq 2$, then the equation $r = r_0 + \delta^{-1}(\partial r + \frac{1}{2h}[r, r])$ has a unique solution belonging to $\Gamma(\Omega_1(M) \otimes W_k)$.*

The contravariant version of Theorem 5.2.2 in [5] is as follows.

Theorem 1 *Let $\bar{\pi} = \sum_{i \geq 1} h^i \pi_i$ be a series of σ-closed 2-vectors on M. The equation*

$$\delta r = -\partial\hat{\pi} + \bar{R} + \partial r + \frac{1}{2h}[r, r] - \bar{\pi}, \quad (5)$$

has a unique solution $r \in \Gamma(\Omega_1(M) \otimes \mathscr{W})$ satisfying the normalization condition $\delta^{-1} r = 0$, and such that the $\mathscr{W}$-degree of the leading term of r is at least 2.

Proof. Let $r_0 = \delta^{-1}(-\partial\hat{\pi} + \bar{R} - \bar{\pi})$, and observe that $\deg r_0 \geq 2$, so from Lemma 2 we deduce the existence of a unique solution r of the equation $r = \delta^{-1}(-\partial\hat{\pi} + \bar{R} - \bar{\pi}) + \delta^{-1}\partial r + \frac{1}{2h}\delta^{-1}[r,r]$. Moreover, since $(\delta^{-1})^2 = 0$ it follows that $\delta^{-1}r = 0$.

In order to prove that r is a solution of Equation (5), we set $K = \delta r + \partial\hat{\pi} - \bar{R} - \partial r - \frac{1}{h}r^2 + \bar{\pi}$. Using the decomposition (3) and the fact that r is a 1-vector (so $r_{00} = 0$), we get $\delta^{-1}K = 0$. Furthermore,

$$DK = \frac{1}{h}[\partial\hat{\pi}, r] + \frac{1}{h}[r, \partial\hat{\pi}] + \partial^2\hat{\pi} - \partial\bar{R} + \delta\bar{R} - \delta\partial\hat{\pi} = -\delta\partial\hat{\pi} = \frac{1}{2h}\partial[\hat{\pi}, \hat{\pi}] = 0.$$

Hence it follows that K can be written as $K = \delta^{-1}(\partial K + \frac{1}{h}[r,K])$. Applying the iteration method to the above equation we can conclude that $K = 0$ and the theorem follows. ■

Since D acts as a derivation the space $\Gamma_D(\mathscr{W}) := \{a \in \Gamma(\mathscr{W}) \mid Da = 0\}$ is a subalgebra of $\Gamma(\mathscr{W})$, and Theorem 5.2.4 in [5] becomes:

Theorem 2 *Given a flat contravariant connection D, for any* $a_0 \in C^\infty(M)[[h]]$ *there is a unique* $a \in \Gamma_D(\mathscr{W})$ *such that* $a(x,0,h) = a_0(x,h)$.

Proof. The proof of the theorem follows exactly the same pattern as the one given in [5]. ■

Finally, using the *symbol map* $\sigma^{-1} : \Gamma_D(\mathscr{W}) \to C^\infty(M)[[h]]$, $\sigma^{-1}(a) = a(x,0,h)$, and the above theorem one deduces the existence of star-product on M by setting

$$a * b = \sigma^{-1}(\sigma(a) \circ \sigma(b)), \qquad a,b \in C^\infty(M)[[h]]. \tag{6}$$

4. Star-products on Abelian Poisson Lie Groups

Definition 5. **([4])** *A Lie group G is called a* Poisson Lie group *if it is also a Poisson manifold such that the multiplication map* $m : G \times G \to G$ *is also a Poisson map, where* $G \times G$ *is equipped with the product Poisson structure.*

In what follows we shall assume that G is a real connected Lie group with Lie algebra $\mathfrak{g}$. Let R_g be the right translation of G by g. We denote by $\pi_r : G \to \wedge^2\mathfrak{g}$ the map defined by $\pi_r(g) = dR_{g^{-1}}\pi(g)$ and by $d_e\pi_r$ its differential at the identity $e \in G$.

Theorem 3. **([4, 9])** *The following conditions are equivalent:*

1. (G, π) *is a Poisson-Lie group.*
2. *The Poisson structure* π *satisfies* $\pi(e) = 0$, *and* $L_X\pi$ *is right invariant, whenever* X *is a right invariant vector field, where* $L_X\pi$ *denotes the Lie derivative of* π *with respect to* X.
3. *The dual map* $(d_e\pi_r)^* : \wedge^2\mathfrak{g}^* \to \mathfrak{g}^*$ *of* $d_e\pi_r$ *defines a Lie algebra structure on* $\mathfrak{g}^*$.

Let G^* be the *dual group* of G, i.e., the connected and simply connected Lie group whose Lie algebra is $\mathbf{g}^*$. Recall that there exists [9] a unique Poisson structure, say π^* on G^* which induces the Lie algebra structure of $\mathbf{g}$.

Next we shall endow the space $G \times G^*$ with a Poisson structure so that we can search for projectable $*$-products under the projection $p_1 : G \times G^* \to G$, $p_1(g, g^*) = g$.

Let $\tilde{X}_i$, $1 \leq i \leq n$, be n right-invariant vector fields on G forming a global frame on it, let $\tilde{X}_{n+i}$ be the right-invariant vector fields on G^* corresponding to the dual basis of $\tilde{X}_i(e)$, and let $p_2 : G \times G^* \to G^*$ be the projection onto the second factor. We denote by X_i, X_{n+i} the $2n$ vector fields on $G \times G^*$ such that $dp_1(X_i) = \tilde{X}_i$, $dp_2(X_i) = 0$, $dp_1(X_{n+i}) = 0$, $dp_2(X_{n+i}) = \tilde{X}_{n+i}$, and by C^k_{ij} (resp. D^{ij}_k) the structure constants of $\mathbf{g}$ (resp. $\mathbf{g}^*$) relative to the basis $\tilde{X}_i(e)$ (resp. $\tilde{X}_{n+i}(e^*)$). The following result was first considered in [7].

Proposition 1 *Let $(G, -\pi)$ be a Lie Poisson group, let $-\pi + \pi^*$ be the product Poisson structure on $G \times G^*$, and let $\psi = \sum_i X_i \wedge X_{n+i}$. Then, the 2-vector $\Lambda = \psi - \pi + \pi^*$ determines a Poisson structure on $G \times G^*$.*

Proof. It is enough to prove that $[\Lambda, \Lambda]_s = 0$. By linearity of the Schouten bracket one gets

$$[\psi - \pi + \pi^*, \psi - \pi + \pi^*]_s = [\psi, \psi]_s - 2[\psi, \pi]_s + 2[\psi, \pi^*]_s,$$

However, writing the right-hand side of the equality relative to the 3-vectors $X_i \wedge X_j \wedge X_k$. $1 \leq i, j, k \leq 2n$, and using condition 2 of Theorem 3 it becomes

$$(C^s_{ia} - L_{X_{n+s}} \pi^*{}_{ia}(e^*)) X_s \wedge X_{n+i} \wedge X_{n+a} + (D^{ia}_s - L_{X_s} \pi^{ia}(e)) X_i \wedge X_a \wedge X_{n+s},$$

which vanishes by virtue of condition 3 of Theorem 3. ■

Remark 1 *The Poisson structure Λ is symplectic in an open neighborhood U of $G \times e^*$ on $G \times G^*$.*

Ideally, we would like to prove the existence of star-products on (G, π), by means of projectable (by p_1) star-products on $G \times G^*$ or U. Unfortunately, we do not know in general if this is possible. However, if we assume that (G, π) is an Abelian Poisson Lie group, then there is a $*$-product on $(G \times G^*, \Lambda)$ such that $(p_1^* C^\infty(G)[[h]], *)$ is a subalgebra of $(C^\infty(G \times G^*)[[h]], *)$. Observe that under the Abelian hypothesis Λ can be written as $\Lambda = \sum_i e_i \wedge e_{n+i}$, where $e_i = X_i$ and $e_{n+i} = X_{n+i} - \frac{1}{2} \sum_j \pi^{ij} X_j$, and so we can define a symplectic contravariant connection by setting

$$\nabla_{\theta^i} e_j = \nabla_{\theta^{n+i}} e_j = 0, \quad \nabla_{\theta^i} e_{n+j} = \nabla_{\theta^{n+i}} e_{n+j} = 0, \tag{7}$$

where $\{\theta^i, \theta^{n+i}\}$ is the dual basis of $\{e_i, e_{n+i}\}$.

Let $\mathscr{W}$ be the Weyl bundle over $G \times G^*$, and let ∂ be the connection on $\mathscr{W}$ induced by ∇. One easily verifies that the local expression of $r = \delta^{-1}(-\partial\hat{\pi}) + \delta^{-1}(\partial r + \frac{1}{h}r^2)$, $\hat{\pi} = e_i \otimes y^i$, in the above basis is of the type

$$r = \sum_{2k+p \geq 2} r_{k,j_1,i_1,\dots,i_p} h^k e_{j_1} \otimes y^{i_1} \cdot \cdot y^{i_p}, \qquad r_{k,j_1,i_1,\dots,i_p} \in p_1^* C^\infty(G). \tag{8}$$

Lemma 3 *Let* $\lambda^i = \theta^i - \frac{1}{2}\sum_j \pi^{ij}\theta^{n+j}$ *for* $1 \leq i \leq n$ *and let* $D = \partial - \delta + \frac{1}{h}[r,\cdot]$. *If* $a_0 \in p_1^* C^\infty(G)[[h]]$, *then* $\partial a_0(\lambda^i) = 0$. *Furthermore, since* ∂ *acts as a derivation, the space* $A = \{a \in \Gamma_D(\mathscr{W}) \mid \partial a(\lambda^i) = 0,\ 1 \leq i \leq n\}$ *is a subalgebra of* $\Gamma_D(\mathscr{W})$.

Theorem 4 *The elements of the subalgebra A are parameterized by* $C^\infty(G)[[h]]$.

Proof. Let $a_0 \in p_1^* C^\infty(G)[[h]]$, and let a be the flat section corresponding to it. Since a is obtained recursively from the equation $a = \delta^{-1}(\partial a + \frac{1}{h}[r,a]) + a_0$, it follows from (8) and the above lemma that $a \in A$. The converse is easily deduced from the fact that $(\lambda^i)^\# = X_{n+i}$. ■

Using a different method, S. Gutt [6] has shown that the star-product constructed on the cotangent bundle of a Lie group G in [2] is projectable to the dual of the Lie algebra of G. Therefore, one can consider Theorem 4 as a generalization of such a result.

Example 1 Let G be the dual of the Heisenberg Lie algebra endowed with the standard Lie-Poisson structure π. Let G^* be its dual group. We define a flat contravariant connection on $G \times G^*$ as in (7) and so a connection ∂ on $\mathscr{W}$. Let (x^1, x^2, x^3) be the coordinate system on G such that $\pi = x^3\partial_1 \wedge \partial_2$. Let r and r' be the respective solutions of the equation of Lemma 2 with $r_0 = -\partial\hat{\pi}$ and $r'_0 = -\partial\hat{\pi} + h\partial_1 \wedge \partial_3$, and let D, D' be the corresponding flat connections. Then one verifies that the two star-products defined on $(G \times G^*, \Lambda)$ corresponding to D and D' are projectable (with respect to the projection onto the first factor) and that the star-products induced on (G, π) are not *equivalent*.

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The deformation quantization of certain super-Poisson brackets and BRST cohomology

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Abstract. On every split supermanifold equipped with the Rothstein super-Poisson bracket we construct a deformation quantization by means of a Fedosov-type procedure. In other words, the supercommutative algebra of all smooth sections of the dual Grassmann algebra bundle of an arbitrarily given vector bundle E (equipped with a fiber metric) over a symplectic manifold M will be deformed by a series of bidifferential operators having first order supercommutator proportional to the Rothstein superbracket. Moreover, we discuss two constructions related to the above result, namely the quantized BRST-cohomology for a locally free Hamiltonian Lie group action and the classical BRST cohomology in the general coisotropic (or reducible) case without using a 'ghosts of ghosts' scheme.

Keywords: Deformation quantization, even symplectic supermanifolds, Fedosov construction, BRST-cohomology, coisotropic submanifolds

Mathematics Subject Classifications (2000): 16S80, 58C50, 81Q60, 81S99

To the memory of Moshé Flato

Introduction

In the usual programme of deformation quantization (cf. [4]) the quantum mechanical multiplication is considered as a formal associative deformation (a so-called star-product) of the pointwise multiplication of the classical observables, viz. the algebra of smooth complex-valued functions on a given symplectic manifold. The deformation is such that to first order in the deformation parameter λ (corresponding to $\hbar$) the commutator of the deformed product is proportional to the Poisson bracket. The difficult question of existence of these star-products for every symplectic manifold was settled independently by De Wilde and Lecomte [14] and Fedosov [17, 18], and even for general Poisson manifolds by Kontsevich [29].

Four years ago, adequate formulations for star-products in the theory of supermanifolds, however, did not seem to have appeared in the literature although the geometric quantization scheme had found its suitable generalization to the super case (see, e.g., [22] and references therein). In order to elaborate our understanding of supermanifolds, (at Freiburg) I proposed to the diploma student Ralf Eckel to give a formulation thereof in terms of associated bundles of certain jet group bundles which he did rather nicely in his diploma thesis [16] in April 1996. He also provided a star-product formula for the case where the 'fermionic directions'

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formed a trivial vector bundle (see below for a precise statement). When preparing my habilitation thesis in May 1996, I suddenly realized that a simple Fedosov procedure could be set up for general vector bundles: However, I did not know in advance a possible super-Poisson bracket, so I first constructed the deformed algebra à la Fedosov and *a posteriori* computed the super-Poisson bracket as its first order supercommutator in [6], a result which I included in my habilitation thesis. A month later I was made aware by Amine El Gradechi that the super-Poisson bracket I had computed out of this quantization exactly coincided with Rothstein's super-Poisson bracket, see [31], found in 1991.

In this report I should like to give a detailed description of this Fedosov construction (thereby including an improved version of the preprint [6] without some rather embarrassing misprints). I shall also include sketches of two more recent constructions related to the above and done in collaboration with H.-C. Herbig and S. Waldmann (see [8, 7]), namely the quantum BRST cohomology for covariant star-products, and the classical BRST cohomology for arbitrary coisotropic constraint surfaces (the 'reducible first-class case' in the physics literature) where a so-called 'ghosts-for-ghosts'-scheme is no longer necessary.

The supermanifolds I shall deal with in this report will only be 'split', more precisely, the set-up will be as follows: Let (M,ω) be a $2m$-dimensional symplectic manifold and E be an arbitrary n-dimensional vector bundle over M. Then the algebra $\mathscr{C}_0$ of 'classical superobservables' can be considered as the space of all smooth sections of the complexified dual Grassmann algebra bundle of E (see, e.g., [3]), i.e.,

$$\mathscr{C}_0 := \mathbb{C}\Gamma(\Lambda E^*), \tag{1}$$

where the multiplication is the pointwise wedge product. Clearly, $\mathscr{C}_0$ is a $\mathbb{Z}_2$-graded supercommutative algebra, i.e., $\phi \wedge \psi = (-1)^{d_1 d_2} \psi \wedge \phi$ for $\phi, \psi \in \Gamma(\Lambda E^*)$, ϕ of degree d_1 and ψ of degree d_2. A *super-Poisson bracket* for $\mathscr{C}_0$ is by definition a $\mathbb{Z}_2$-graded bilinear map $M_1 : \mathscr{C}_0 \times \mathscr{C}_0 \to \mathscr{C}_0$ which is superanticommutative, i.e., $M_1(\psi,\phi) = -(-1)^{d_1 d_2} M_1(\phi,\psi)$, satisfies the superderivation rule $M_1(\phi, \psi \wedge \chi) = M_1(\phi,\psi) \wedge \chi + (-1)^{d_1 d_2} \psi \wedge M_1(\phi,\chi)$, and the super-Jacobi identity, i.e., $(-1)^{d_1 d_3} M_1(M_1(\phi,\psi),\chi) + cycl. = 0$ where $\chi \in \mathscr{C}_0$ is of degree d_3.

It is in general not difficult to find super-Poisson brackets of purely algebraic type, i.e., which vanish when one of their arguments is restricted to a smooth complex-valued function, by means of a fiber metric q in E (see, e.g., [4], p. 123, eq. 5-1):

$$M_1'(\phi,\psi) = q^{AB} (j(e_A)\phi) \wedge (i(e_B)\psi), \tag{2}$$

where q^{AB} are the components of the induced fiber metric in the dual bundle E^* in the dual basis to a local basis (e_A), $1 \leq A \leq \dim E$, of sections of E, and $i(e_B)$ and $j(e_A)$ denote the usual interior product left antiderivation and right antiderivation, respectively, which are also often denoted by $\overrightarrow{\partial}_A$ and $\overleftarrow{\partial}_A$ in the literature on supermanifolds. The above definition does not depend on the choice of that local basis. In the case where M is $\mathbb{R}^{2m}$ with the standard Poisson bracket one can combine the standard bracket with the above super-Poisson bracket to get

$$M_1(\phi,\psi) = \frac{\partial\phi}{\partial q^i}\wedge\frac{\partial\psi}{\partial p_i} - \frac{\partial\phi}{\partial p_i}\wedge\frac{\partial\psi}{\partial q^i} + q^{AB}(j(e_A)\phi)\wedge(i(e_B)\psi). \tag{3}$$

However, for nontrivial bundles it does not seem to be so obvious to generalize this bracket in the sense that it is equal to — at least in degree zero — the Poisson bracket of the base space M when restricted to the sections of degree zero. Some time ago M. Rothstein has given a formula for this more general situation, [31]:

$$\{\phi,\psi\}_R = \Lambda^{ij}((1-2\hat{R}^E)^{-1})_i^k \wedge \nabla^E_{\partial_k}\phi \wedge \nabla^E_{\partial_j}\psi + q^{AB}(j(e_A)(\phi))\wedge(i(e_B)(\psi)), \tag{4}$$

where ∇^E is a covariant derivative in the bundle E preserving the fiber metric q and $\hat{R}^E$ is a suitable section in the bundle $\Gamma(Hom(TM,TM)\otimes\Lambda^{\mathrm{even}}E^*)$ constructed out of the curvature of ∇^E (see Section 1 for details).

The paper is organized as follows: In the first part, I transfer Fedosov's Weyl algebra bundle to the above situation by simply tensoring with the dual Grassmann bundle ΛE^*. The fiberwise multiplication has also a component in ΛE^* built by means of the fiber metric in E. Then Fedosov's procedure can completely be imitated without further difficulties: We show the existence of a Fedosov connection D of square zero whose kernel in the space of antisymmetric degree zero is in linear 1-1 correspondence to the space of formal power series in λ with coefficients in $\mathscr{C}_0$

$$\mathscr{C} := \mathscr{C}_0[[\lambda]], \tag{5}$$

which immediately gives rise to the desired quantum deformation (Theorem 1.3). Then I explicitly compute the super-Poisson bracket M_1 as the term proportional to $(\mathbf{i}\lambda)/2$ by means of Fedosov's recursion formulas (Theorem 1.4) and show that it is equal to the Rothstein superbracket. We evaluate the formulas a little bit further in Section 2 in the case where the connection ∇^E is flat: Using a local basis of covariantly constant sections the quantum multiplication is a sort of tensor product of a star-product on (the smooth complex-valued functions on) M and a formal Clifford multiplication. Section 3 is concerned with a sketch of a quantized BRST formulation (see [8]). In Section 4 I shall sketch joint work with Hans-Christian Herbig where we have found a classical BRST complex for general coisotropic (reducible first class) constraint manifolds using the Rothstein superbracket, see also [7].

Notation: In all of this paper the Einstein summation convention is used: Two equal indices are automatically summed up over their range unless stated otherwise. Moreover, we widely make use of Fedosov's notation in [18] with the following exceptions: We use the symbol ∇ to denote the covariant derivative and not Fedosov's ∂ and describe the occurring symmetric tensor fields with $\vee$ products (see, e.g., [23], p. 209–226) and use the symmetric substitution operator i_s instead of Fedosov's functions of y and derivatives with respect to y.

1. A star-product for sections of Grassmann algebra bundles

This Section is — up to some corrected typos and an additional remark — identical to [6].

1.1. The Fedosov construction

Let (M,ω) be a $2m$-dimensional symplectic manifold and E an n-dimensional real vector bundle over M with a fixed nondegenerate fiber metric q. For the computations that will follow we shall use coordinates $(x^1,\cdots,x^{2m})$ in a chart U of M. The base fields $\frac{\partial}{\partial x^i}$ will be denoted by ∂_i $(1\leq i\leq 2m)$ for short. For computations in E we shall use a local basis (e_A), $1\leq A\leq n$, of sections of E over U. Denote the dual basis in the dual bundle E^* of E by (e^A), $(1\leq A\leq n)$. Let $\Lambda\in\Gamma(\Lambda^2 TM)$ denote the Poisson structure of (M,ω), i.e., the Poisson bracket of two smooth real valued functions f,g is given by $\{f,g\}:=\Lambda(df,dg)$. Denoting the components of ω and Λ in that chart by $\omega_{ij}:=\omega(\partial_i,\partial_j)$ and $\Lambda^{ij}:=\Lambda(dx^i,dx^j)$ we use the sign conventions of [1] where $\Lambda^{ik}\omega_{jk}=\delta^i_j$. Fix a torsion-free symplectic connection ∇^M in the tangent bundle of M. This is well known to always exist which can be seen by Heß's formula $\omega(\nabla^M_X Y,Z):=\omega(\tilde{\nabla}_X Y,Z)+\frac{1}{3}(\tilde{\nabla}_X\omega)(Y,Z)+\frac{1}{3}(\tilde{\nabla}_Y\omega)(X,Z)$ where X,Y,Z are arbitrary vector fields on M and $\tilde{\nabla}$ is an arbitrary torsion-free connection on M (see [25]). Fix a connection ∇^E in E which is compatible with q, i.e., $X(q(e_1,e_2))=q(\nabla^E_X e_1,e_2)+q(e_1,\nabla^E_X e_2)$ for an arbitrary vector field X on M and sections e_1,e_2 of E. This is also well known to always exist which can be seen by the formula $q(\nabla^E_X e_1,e_2):=q(\tilde{\nabla}^E_X e_1,e_2)+\frac{1}{2}(\tilde{\nabla}^E_X q)(e_1,e_2)$ for an arbitrary connection $\tilde{\nabla}^E$ in E.

We are now forming the *Fedosov algebra* $\mathscr{W}\otimes\Lambda$:

$$\mathscr{W}\otimes\Lambda:=\big(\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M\otimes\Lambda E^*\otimes\Lambda T^*M))\big)[[\lambda]]. \tag{6}$$

This is to say that the elements of $\mathscr{W}\otimes\Lambda$ are formal sums $\sum_{s,t=0}^{\infty}w_{st}\lambda^t$ where the w_{st} are smooth sections in the complexification of the vector bundle $\vee^s T^*M\otimes\Lambda E^*\otimes\Lambda T^*M$. In what follows we shall sometimes use the following factorized sections $F:=f\otimes\phi\otimes\alpha\lambda^{t_1}$ and $G:=g\otimes\psi\otimes\beta\lambda^{t_2}$ where $f\in\Gamma(\vee^{s_1}T^*M)$, $g\in\Gamma(\vee^{s_2}T^*M)$, $\phi\in\Gamma(\Lambda^{d_1}E^*)$, $\psi\in\Gamma(\Lambda^{d_2}E^*)$, $\alpha\in\Gamma(\Lambda^{a_1}T^*M)$, and $\beta\in\Gamma(\Lambda^{a_2}T^*M)$. Let $deg_s, deg_E, deg_a, deg_\lambda$ be the obvious degree maps from $\mathscr{W}\otimes\Lambda$ to itself, i.e., those $\mathbb{C}$-linear maps for which the above factorized sections $f\otimes\phi\otimes\alpha\lambda^{t_1}$ are eigenvectors to the eigenvalues s_1,d_1,a_1,t_1 respectively and which we refer to as the symmetric degree, the E-degree, the antisymmetric degree, and the λ-degree, respectively. Moreover, let P_E and P_λ be the corresponding maps $(-1)^{deg_E}$ and $(-1)^{deg_\lambda}$ which we refer to as the E-parity and the λ-parity, respectively. We say that a $\mathbb{C}$-linear endomorphism Φ of $\mathscr{W}\otimes\Lambda$ is of ζ-degree k $(\zeta=s,a,E,\lambda)$ iff $[deg_\zeta,\Phi]=k\Phi$. Analogously, Φ is said to be of ζ-parity $(-1)^k$ $(\zeta=E,\lambda)$ iff $P_\zeta\Phi P_\zeta=(-1)^k\Phi$. Let C denote the complex conjugation of sections in $\mathscr{W}\otimes\Lambda$. We shall sometimes write $\mathscr{W}$ for the space of elements of $\mathscr{W}\otimes\Lambda$ having zero antisymmetric degree and $\mathscr{W} \otimes \Lambda^a$ for the space of those elements having antisymmetric degree a. The space $\mathscr{W} \otimes \Lambda$ is an associative algebra with respect to the usual pointwise product where we do *not* use the graded tensor product of the two Grassmann algebras involved. More precisely, for the above factorized sections the pointwise or undeformed multiplication is simply given by

$$(f \otimes \phi \otimes \alpha\lambda^{t_1})(g \otimes \psi \otimes \beta\lambda^{t_2}) := (f \vee g) \otimes (\phi \wedge \psi) \otimes (\alpha \wedge \beta)\lambda^{t_1+t_2}. \tag{7}$$

Note that the above four degree maps are derivations and the above two parity maps are automorphisms of this multiplication. Moreover, $\mathscr{W} \otimes \Lambda$ is supercommutative in the sense that

$$GF = (-1)^{d_1 d_2 + a_1 a_2} FG. \tag{8}$$

A linear endomorphism Φ of $\mathscr{W} \otimes \Lambda$ of E-parity $(-1)^{d'}$ and antisymmetric degree a' is said to be a superderivation of type $((-1)^{d'}, a')$ of the undeformed algebra $\mathscr{W} \otimes \Lambda$ iff $\Phi(FG) = (\Phi F)G + (-1)^{d'd_1 + a'a_1} F(\Phi G)$. Let σ denote the linear map

$$\sigma : \mathscr{W} \otimes \Lambda \to \Gamma(\Lambda E^* \otimes \Lambda T^* M)[[\lambda]] \tag{9}$$

which projects onto the component of symmetric degree zero and clearly is a homomorphism for the undeformed multiplication.

We now combine the two covariant derivatives ∇^M_X in TM and ∇^E_X in E into a covariant derivative ∇_X in $TM \otimes E$ in the usual fashion and extend it canonically to a connection ∇ in $\mathscr{W} \otimes \Lambda$. On the above factorized sections we get in a chart:

$$\nabla(f \otimes \phi \otimes \alpha) = \left((\nabla^M_{\partial_i} f) \otimes \phi + f \otimes (\nabla^E_{\partial_i} \phi)\right) \otimes (dx^i \wedge \alpha) + f \otimes \phi \otimes d\alpha. \tag{10}$$

In order to define a deformed fiberwise associative multiplication consider the following insertion maps for a vector field X on M and a section e of E: Let $i_a(X)$ and $i(e)$ denote the usual inner product antiderivations in $\Gamma(\Lambda T^* M)$ and $\Gamma(\Lambda E^*)$, respectively, and extend them in a canonical manner to superderivations of type $(1, -1)$ and $(-1, 0)$ of the undeformed algebra $\mathscr{W} \otimes \Lambda$, respectively. Let $j(e)$ be defined by $P_E i(e)$. Moreover, let $i_s(X)$ denote the corresponding inner product derivation (or symmetric substitution, [23], p. 209–226) in $\times_{s=0}^{\infty} \Gamma(\vee^s T^* M)$, again extended to a derivation of the undeformed algebra $\mathscr{W} \otimes \Lambda$ in the canonical way. Let q^{AB} denote the components of the induced fiber metric q^{-1} in E^*, i.e., $q^{AB} := q^{-1}(e^A, e^B)$. Note that q^{AB} is the inverse matrix to $q(e_A, e_B)$. Then for two elements F, G of $\mathscr{W} \otimes \Lambda$ we can now define the fiberwise deformed multiplication $\circ$:

$$\begin{aligned} F \circ G := \sum_{k,l=0}^{\infty} \frac{(\mathbf{i}\lambda/2)^{k+l}}{k!\,l!} \Lambda^{i_1 j_1} \cdots \Lambda^{i_k j_k} q^{A_1 B_1} \cdots q^{A_l B_l} \times \\ \times \left(i_s(\partial_{i_1}) \cdots i_s(\partial_{i_k}) j(e_{A_1}) \cdots j(e_{A_l}) F\right)\left(i_s(\partial_{j_1}) \cdots i_s(\partial_{j_k}) i(e_{B_1}) \cdots i(e_{B_l}) G\right). \end{aligned} \tag{11}$$

Moreover, let δ and δ^* be the canonical superderivations of the undeformed algebra $\mathscr{W} \otimes \Lambda$ of type $(1, 1)$ and $(1, -1)$, respectively, which are induced by the identity map of T^*M to T^*M, where in the case of δ the preimage of the identity is regarded as being part of $\vee T^*M$ and the image as being part of ΛT^*M, and vice versa in the case of δ^*. On the above factorized sections these maps read in coordinates

$$\begin{aligned} \delta(f \otimes \phi \otimes \alpha) &= (i_s(\partial_i)f) \otimes \phi \otimes (dx^i \wedge \alpha), && (12) \\ \delta^*(f \otimes \phi \otimes \alpha) &= (dx^i \vee f) \otimes \phi \otimes (i_a(\partial_i)\alpha). && (13) \end{aligned}$$

Define the total degree derivation *Deg*:

$$Deg := 2deg_\lambda + deg_s + deg_E. \tag{14}$$

A $\circ$-superderivation of type $((-1)^{d'}, a')$ is defined in an analogous manner as for the undeformed multiplication. Note that $(\mathscr{W} \otimes \Lambda, \circ)$ is not a graded associative algebra in the strict sense since it is equal to the Cartesian product of the eigenspaces of *Deg* and not to the direct sum of these eigenspaces. It is, however, filtered by those complex subspaces of $\mathscr{W} \otimes \Lambda$ (indexed by a nonnegative integer r) which are given by the images of the maps $Deg(Deg-1)\cdots(Deg-(r-1))$.

We collect some properties of the above structures in the following Proposition:

Proposition 1.1 *With the above definitions and notations we have the following:*

1. $\delta^2 = 0 = (\delta^*)^2$ *and* $\delta\delta^* + \delta^*\delta = deg_s + deg_a$.
2. $\delta\nabla + \nabla\delta = 0$.
3. $Ker(\delta) \cap Ker(deg_a) = \mathscr{C}$.
4. P_E *is a* $\circ$*-automorphism and* deg_a *is a* $\circ$*-derivation which equips the Fedosov algebra* $(\mathscr{W} \otimes \Lambda, \circ)$ *with the structure of a* $\mathbb{Z}_2 \times \mathbb{Z}$*-graded associative algebra.*
5. δ, ∇, *and Deg are* $\circ$*-superderivations of type* $(1,1)$, $(1,1)$, *and* $(1,0)$, *respectively.*
6. *The parity map* P_λ *and the complex conjugation C are graded* $\circ$*-antiautomorphisms, i.e.,* $\Phi(F \circ G) = (-1)^{d_1 d_2 + a_1 a_2}\Phi(G) \circ \Phi(F)$ *for* $\Phi = P_\lambda, C$.

Proof. 1. Straightforward.
2. This follows from the vanishing torsion of ∇^M.
3. Without the factor ΛE^* the kernel of δ in the space of antisymmetric degree zero consists of the constants, which proves this statement.
4. The associativity of $\circ$ is known, see, e.g., [4], p. 123, eq. 5-2, and can be done by a long straightforward computation. We shall sketch a shorter proof: $\circ$ is defined on each fiber (for $m \in M$) $\mathscr{W}_m := (\times_{i=0}^{\infty}(\vee^i T_m M^* \otimes \Lambda E_m^* \otimes \Lambda T_m M^*))[[\lambda]]$ on which we can rewrite the multiplication in the more compact form ($F, G \in \mathscr{W}_m$)

$$F \circ G = \mu(e^{\frac{i\lambda}{2}(R+S)}(F \otimes G)), \tag{15}$$

where the tensor product is over $\mathbb{C}[[\lambda]]$, μ denotes the undeformed fiberwise multiplication, and $R := \Lambda^{ij}_m i_s(\partial_i)\otimes i_s(\partial_j)$, $S := q^{AB}_m j(e_A)\otimes i(e_B)$. Due to the derivation properties of $i_s(\partial_i)$, $i(e_A)$, and $j(e_B)$ we get formulas like

$$\begin{aligned} R\,\mu\otimes 1 &= \mu\otimes 1\,(R_{13}+R_{23}),\\ R\,1\otimes\mu &= 1\otimes\mu\,(R_{12}+R_{13}),\\ S\,\mu\otimes 1 &= \mu\otimes 1\,(S_{13}(P_E)_2+S_{23}),\\ S\,1\otimes\mu &= 1\otimes\mu\,(S_{12}+S_{13}(P_E)_2). \end{aligned}$$

where the index notation is borrowed from Hopf algebras and indicates on which of the three tensor factors of $\mathscr{W}_m$ the maps R, S, and P_E should act, e.g., $R_{23} := 1\otimes R$, $(P_E)_2 := 1\otimes P_E\otimes 1$. These "pull through formulas" can be used to pull through the corresponding formal exponentials. Since all the maps $i_s(\partial_i)$ commute with $j(e_A)$ and $i(e_B)$ and since the $j(e_A)$ commute with all $i(e_B)$ whereas $j(e_A)$ and $j(e_B)$ anticommute as well as $i(e_A)$ and $i(e_B)$ we can conclude that all the six maps R_{12}, R_{13}, R_{23}, S_{12}, $S_{13}(P_E)_2$, and S_{23} pairwise commute. This is the essential step for associativity. The gradation properties are immediate.
5. The derivation properties of δ and Deg are clear, for the corresponding statement for ∇ the fact that ∇^M preserves the Poisson structure Λ and that ∇^E preserves the dual fiber metric q^{-1} is crucial.
6. Straightforward. ■

Due to the first part of this proposition we can construct a $\mathbb{C}[[\lambda]]$-linear endomorphism δ^{-1} of the Fedosov algebra in the following way: On the above factorized sections F we put

$$\delta^{-1}F := \begin{cases} \frac{1}{s_1+a_1}\delta^* F & \text{if } s_1+a_1\geq 1\\ 0 & \text{if } s_1+a_1=0 \end{cases}. \tag{16}$$

Since $\mathscr{W}\otimes\Lambda$ is an $\mathbb{Z}_2\times\mathbb{Z}$-graded associative algebra we can form the $\mathbb{Z}_2\times\mathbb{Z}$-graded super Lie bracket which reads on the above factorized sections:

$$[F,G] := ad(F)G := F\circ G-(-1)^{d_1d_2+a_1a_2}G\circ F. \tag{17}$$

It follows from the associativity of $\circ$ that $ad(F)$ is $\circ$-superderivation of the Fedosov algebra $(\mathscr{W}\otimes\Lambda,\circ)$ of type $((-1)^{d_1},a_1)$. Note that the map $\frac{\mathrm{i}}{\lambda}ad(F)$ which we shall often use in what follows is always well defined because of the supercommutativity of the undeformed multiplication (8).
Consider now the curvature tensors R^M of ∇^M and R^E of ∇^E, i.e., for three vector fields X,Y,Z on M and a section e of E we have $R^M(X,Y)Z = \nabla^M_X\nabla^M_Y Z - \nabla^M_Y\nabla^M_X Z - \nabla^M_{[X,Y]}Z$ and $R^E(X,Y)e = \nabla^E_X\nabla^E_Y e - \nabla^E_Y\nabla^E_X e - \nabla^E_{[X,Y]}e$. Define elements $R^{(M)}$ and $R^{(E)}$ of the Fedosov algebra which are contained in $\Gamma(\vee^2T^*M\otimes\Lambda^2T^*M)$

and $\Gamma(\Lambda^2 E^* \otimes \Lambda^2 T^* M)$, respectively, as follows where V, W are vector fields on M and e_1, e_2 are sections of E:

$$R^{(M)}(V,W,X,Y) \;:=\; \omega(V,R^M(X,Y)W), \tag{18}$$

$$R^{(E)}(e_1,e_2,X,Y) \;:=\; -q(e_1,R^E(X,Y)e_2). \tag{19}$$

Note that this is well defined: Since ∇^M preserves ω and ∇^E preserves q it follows that $R^{(M)}$ is symmetric in V, W and $R^{(E)}$ is antisymmetric in e_1, e_2. In coordinates these two elements of the Fedosov algebra can be written in the form $R^{(M)} = (1/4)R^{(M)}_{klij}dx^k \vee dx^l \otimes 1 \otimes dx^i \wedge dx^j$ and $R^{(E)} = (1/4)R^{(E)}_{ABij} 1 \otimes e^A \wedge e^B \otimes dx^i \wedge dx^j$. Set

$$R := R^{(M)} + R^{(E)}. \tag{20}$$

Then the following Proposition is immediate:

Proposition 1.2 *With the above definitions and notations we have:*

1. $\nabla^2 = \frac{\mathrm{i}}{\lambda} ad(R)$.
2. $P_E(R) = R$, $P_\lambda(R) = R$ *and* $C(R) = R$.
3. $\delta R = 0$.
4. $\nabla R = 0$.

Proof. 1. Straightforward computation.
2. Obvious.
3. This is a consequence of the vanishing torsion of ∇^M (first Bianchi identity).
4. This is a reformulation of the second Bianchi identity for linear connections in arbitrary vector bundles. ■

We shall now make the ansatz for a Fedosov connection D, i.e., we are looking for an element $r \in \mathscr{W} \otimes \Lambda^1$ *of even E-parity*, i.e., $P_E(r) = r$, such that the map

$$D := -\delta + \nabla + \frac{\mathrm{i}}{\lambda} ad(r) \tag{21}$$

has square zero, i.e., $D^2 = 0$. The following properties of D for any r are crucial:

Lemma 1.1 *Let r be an arbitrary element of $\mathscr{W} \otimes \Lambda^1$ of even E-parity. Then*

1. $D^2 = \frac{\mathrm{i}}{\lambda} ad(-\delta r + \nabla r + R + \frac{\mathrm{i}}{\lambda} r \circ r)$.
2. $D(-\delta r + \nabla r + R + \frac{\mathrm{i}}{\lambda} r \circ r) = 0$.

Proof. This is straightforward using Proposition 1.2 and the fact that $r \circ r = \frac{1}{2}[r,r]$ for the above elements r of even E-parity and odd antisymmetric degree. ■

For an arbitrary element $w \in \mathscr{W} \otimes \Lambda$ we shall make the following decomposition according to the total degree Deg:

$$w = \sum_{k=0}^{\infty} w^{(k)} \quad \text{where } Deg(w^{(k)}) = kw^{(k)}. \tag{22}$$

Note that each $w^{(k)}$ is always a *finite* sum of sections in some $\Gamma(\vee^s T^*M \otimes \Lambda E^* \otimes \Lambda T^*M)$. The subspaces of all elements of $\mathscr{W} \otimes \Lambda$, $\mathscr{W}$, $\mathscr{W} \otimes \Lambda^a$, and $\mathscr{C}$ of total degree k will be denoted by $\mathscr{W}^{(k)} \otimes \Lambda$, $\mathscr{W}^{(k)}$, $\mathscr{W}^{(k)} \otimes \Lambda^a$, and $\mathscr{C}^{(k)}$, respectively.

As in Fedosov's paper [18] there is the following Theorem:

Theorem 1.1 *With the above definitions and notations: Let $r \in \mathscr{W} \otimes \Lambda^1$ be defined by the following recursion:*

$$\begin{aligned} r^{(3)} &:= \delta^{-1} R, \\ r^{(k+3)} &:= \delta^{-1} \left(\nabla r^{(k+2)} + \frac{\mathbf{i}}{\lambda} \sum_{l=1}^{k-1} r^{(l+2)} \circ r^{(k-l+2)} \right). \end{aligned}$$

Then r has the following properties: It is real ($C(r) = r$), depends only on λ^2 ($P_\lambda(r) = r$), has even E-parity, and is in the kernel of δ^{-1}. Moreover, the corresponding Fedosov derivation $D = -\delta + \nabla + (\mathbf{i}/\lambda)ad(r)$ has square zero.

Proof. The behavior of r under the parity transformations and complex conjugation immediately follows from the fact that they commute with δ^{-1} and from their (anti)homomorphism properties (Prop.1.1, 3., 5.; Prop.1.2, 2.).
Let $A := -\delta r + \nabla r + R + \frac{\mathbf{i}}{\lambda} r \circ r =: -\delta r + R + B$. Recall the equation $\delta\delta^{-1} + \delta^{-1}\delta = 1$ on the subspace of the Fedosov algebra where $deg_s + deg_a$ have nonzero eigenvalues. Clearly, $A^{(2)} = -\delta r^{(3)} + R = 0$ because $\delta R = 0$ (Prop.1.2,3.) hence $R = \delta\delta^{-1}R$. Suppose $A^{(l)} = 0$ for all $2 \leq l \leq k+1$. By Lemma 1.1, 2. we have $0 = (DA)^{(k+1)} = -\delta A^{(k+2)} = -\delta B^{(k+2)}$. Hence $B^{(k+2)} = \delta\delta^{-1}B^{(k+2)} = \delta r^{(k+3)}$ proving $A^{k+2} = 0$ which inductively implies $D^2 = 0$ since we had already shown that r is of even E-parity. ■

We shall now compute the kernel of the Fedosov derivation. More precisely, define

$$\mathscr{W}_D := Ker(D) \cap Ker(deg_a). \tag{23}$$

As in Fedosov's paper [18] we have the important characterization:

Theorem 1.2 *With the above definitions and notations: $\mathscr{W}_D$ is a subalgebra of the Fedosov algebra $(\mathscr{W} \otimes \Lambda, \circ)$. Moreover, the map σ (9) restricted to $\mathscr{W}_D$ is a $\mathbb{C}[[\lambda]]$-linear bijection onto $\mathscr{C}$.*

Proof. The kernel of a superderivation is always a subalgebra. Since D and σ are $\mathbb{C}[[\lambda]]$-linear the subalgebra $\mathscr{W}_D$ is a $\mathbb{C}[[\lambda]]$-submodule of $\mathscr{W}$.
Let $w \in \mathscr{W}$. Decompose $w = w_0 + w_+$ where $w_0 := \sigma(w)$ and $w_+ := (1-\sigma)(w)$. We shall prove by induction over the total degree k that $w \in \mathscr{W}$ is in $\mathscr{W}_D$ iff for all nonnegative integers k $w_0^{(k)}$ is arbitrary in $\mathscr{C}^{(k)}$ and $w_+^{(k)}$ is uniquely given by the equation

$$w_+^{(k)} = \delta^{-1}\left(\nabla w^{(k-1)} + \frac{\mathrm{i}}{\lambda}\sum_{l=1}^{k-2}\left[r^{(l+2)}, w^{(k-1-l)}\right]\right) =: \delta^{-1}((Aw)^{(k-1)}), \tag{24}$$

where of course an empty sum is defined to be zero and $w_+^{(0)} = 0$. Note that $Dw = -\delta w + Aw$ and that the $\mathbb{C}$-linear map A does not lower the total degree of w.

Now the equation $(Dw)^{(k)} = 0$ is equivalent to the inhomogeneous equation $\delta(w^{(k+1)}) = (Aw)^{(k)}$. A necessary condition for this equation to be solvable for $w^{(k+1)}$ clearly is $\delta((Aw)^{(k)}) = 0$. But this is also sufficient since then $(Aw)^{(k)} = \delta\delta^{-1}(Aw)^{(k)}$ and we have the particular solution $w_+^{(k+1)} = \delta^{-1}(Aw)^{(k)}$ (since $\sigma\delta^{-1} = 0$) which satisfies (24). To this particular solution any solution to the homogeneous equation $\delta(w^{(k+1)}) = 0$ can be added which precisely is the space $\mathscr{C}^{(k+1)}$.

It remains to show that conversely every initial piece $w' := w_0^{(0)} + w_0^{(1)} + w_+^{(1)} + \cdots + w_0^{(k)} + w_+^{(k)}$ where $w_0^{(l)}$ was arbitrarily chosen in $\mathscr{C}^{(l)}$, $w_+^{(l)}$ is determined by (24) for all $0 \leq l \leq k$, and $(Dw')^{(l)} = 0$ for all $-1 \leq l \leq k-1$ can be continued to $w'' := w' + w_0^{(k+1)} + w_+^{(k+1)}$ with $w_0^{(k+1)}$ arbitrary in $\mathscr{C}^{(k+1)}$, $w_+^{(k+1)}$ determined by (24), and $(Dw'')^{(k)} = 0$. By induction, this will eventually lead to $w \in \mathscr{W}_D$ characterized by the above properties. Indeed, since $D^2 = 0$ we have $0 = (D^2w')^{(k-1)} = -\delta((Dw')^{(k)}) = -\delta((Aw')^{(k)})$. Define $w_+^{(k+1)}$ by $\delta^{-1}((Aw')^{(k)})$ and choose any $w_0^{(k+1)} \in \mathscr{C}^{(k+1)}$. It follows at once that $w_+^{(k+1)}$ satisfies (24) and that we get $(Dw'')^{(k)} = 0$ which proves the induction and the Theorem. ■

Let

$$\tau : \mathscr{C} \to \mathscr{W}_D \subset \mathscr{W} \tag{25}$$

be the inverse of the restriction of σ to $\mathscr{W}_D$. For $\phi \in \Gamma(\Lambda E^*)$ we shall speak of $\tau(\phi)$ as the *Fedosov-Taylor series of* ϕ and refer to the components $\tau(\phi)^{(k)}$ as the Fedosov-Taylor coefficients. We collect some of the properties of τ in the following Proposition:

Proposition 1.3 *With the above definitions and notations:*

1. *τ commutes with P_E, P_λ, and C.*

2. *Let* $\phi = \sum_{d=0}^{n} \phi^{(d)} \in \Gamma(\Lambda E^*)$ *where* $n := \dim E$. *Then* $Deg(\phi^{(d)}) = d\phi^{(d)} = deg_E(\phi^{(d)})$. *Moreover*

$$\tau(\phi)^{(0)} = \phi^{(0)}, \tag{26}$$

$$\tau(\phi)^{(1)} = \delta^{-1}(\nabla\phi^{(0)}) + \phi^{(1)}, \tag{27}$$

$$\vdots \qquad \vdots$$

$$\tau(\phi)^{(n)} = \delta^{-1}\left(\nabla(\tau(\phi)^{(n-1)}) + \frac{\mathbf{i}}{\lambda}\sum_{l=1}^{n-2}\left[r^{(l+2)}, \tau(\phi)^{(n-1-l)}\right]\right) + \phi^{(n)}, \tag{28}$$

$$\tau(\phi)^{(n+1)} = \delta^{-1}\left(\nabla(\tau(\phi)^{(n)}) + \frac{\mathbf{i}}{\lambda}\sum_{l=1}^{n-1}\left[r^{(l+2)}, \tau(\phi)^{(n-l)}\right]\right), \tag{29}$$

$$\vdots \qquad \vdots$$

$$\tau(\phi)^{(k+1)} = \delta^{-1}\left(\nabla(\tau(\phi)^{(k)}) + \frac{\mathbf{i}}{\lambda}\sum_{l=1}^{k-1}\left[r^{(l+2)}, \tau(\phi)^{(k-l)}\right]\right), \tag{30}$$

where $k \geq n$. *The Fedosov-Taylor series* $\tau(\phi)$ *depends only on* λ^2.

3. *For any nonnegative integer k the map* $\phi \mapsto \tau(\phi)^{(k)}$ *is a polynomial in* λ *whose coefficients are differential operators from* $\Gamma(\Lambda E^*)$ *into some* $\Gamma(\vee^s T^*M \otimes \Lambda E^*)$ *of order k.*

Proof. Since r is invariant under the parity maps and complex conjugation, it follows that D commutes with these three maps, hence $\mathscr{W}_D$ is stable under these maps. Since σ obviously commute with them, so does the inverse of its restriction to $\mathscr{W}_D$, τ. The rest is a consequence of the preceding Theorem and a straightforward induction. ■

Define the following $\mathbb{C}[[\lambda]]$-bilinear multiplication on $\mathscr{C}$: for $\phi, \psi \in \mathscr{C}$

$$\phi * \psi := \sigma(\tau(\phi) \circ \tau(\psi)). \tag{31}$$

We shall call $*$ the *Fedosov star-product associated to* $(M, \omega, \nabla^M, E, q, \nabla^E)$. For $\phi, \psi \in \Gamma(\Lambda E^*)$ the star-product $\phi * \psi$ will be a formal power series in λ which we shall write in the following form:

$$\phi * \psi =: \sum_{t=0}^{\infty}\left(\frac{\mathbf{i}\lambda}{2}\right)^t M_t(\phi, \psi). \tag{32}$$

We list some important properties of the Fedosov star-product in the following

Theorem 1.3 *With the above definitions and notations:*

1. *The Fedosov star-product is associative and* $\mathbb{Z}_2$*-graded, i.e.,* P_E *is an automorphism of* $(\mathscr{C},*)$. *The map* P_λ *and the complex conjugation* C *are graded antiautomorphisms of* $(\mathscr{C},*)$.

2. *The* $\mathbb{C}$*-bilinear maps* M_t *are all bidifferential, real, vanish on the constant functions in each argument for* $t \geq 1$, *and have the following symmetry property:*
$$M_t(\psi,\phi) = (-1)^t(-1)^{d_1d_2}M_t(\phi,\psi). \tag{33}$$

3. *The term of order* 0 *is equal to the pointwise Grassmann multiplication. Hence* $(\mathscr{C},*)$ *is a formal associative deformation of the supercommutative algebra* $(\mathscr{C}_0,\wedge)$.

Proof. Basically, every stated property is easily derived from the definitions (31) and (32) and the corresponding behavior of the fiberwise multiplication $\circ$ under P_E, P_λ, and C. The reality of the M_t follows easily from the graded antihomomorphism property of C once Equation (33) is proved by means of the graded antihomomorphism property of the λ-parity. Since $\tau(1)$ is easily seen to be equal to 1 we have $1 * \psi = \psi = \psi * 1$, and the M_t must vanish on 1 for $t \geq 1$. Finally, each M_t obviously depends on only a finite number of Fedosov-Taylor coefficients whence it must be bidifferential. ■

1.2. Computation of the super-Poisson bracket

In this section we are going to compute an explicit expression for the term M_1 of the Fedosov star-product defined in the last section (compare (32) and Theorem 1.3). Only by means of the graded associativity of the deformed algebra $(\mathscr{C},*)$ we can derive the following Lemma:

Lemma 1.2 *Let* ϕ,ψ,χ *be sections in* $\mathscr{C}_0$ *of* E*-degree* d_1,d_2,d_3, *respectively. Then*

$$\begin{aligned} M_1(\psi,\phi) &= -(-1)^{d_1d_2}M_1(\phi,\psi), && (34)\\ M_1(\phi,\psi\wedge\chi) &= M_1(\phi,\psi)\wedge\chi+(-1)^{d_1d_2}\psi\wedge M_1(\phi,\chi), && (35)\\ 0 &= (-1)^{d_1d_3}M_1(M_1(\phi,\psi),\chi)+\text{cycl.} && (36)\end{aligned}$$

Hence M_1 *is a* super-Poisson bracket *on* $\mathscr{C}_0$.

Proof. The first property is a particular case of (33). Consider now the graded commutator $[\phi,\psi] := \phi*\psi-(-1)^{d_1d_2}\psi*\phi$ on $\mathscr{C}$. Because of the graded associativity of $*$ we have the superderivation property $[\phi,\psi*\chi] = [\phi,\psi]*\chi+(-1)^{d_1d_2}\psi*[\phi,\chi]$. Writing this out with the M_t and taking the term of order λ we get the second property. For the third, take the term of order λ^2 in the super-Jacobi identity for the graded commutator. ■

Before we are going to compute M_1 directly it is useful to introduce the following notions:
For ϕ in $\mathscr{C}_0$ let ϕ_1 and ρ denote the component of symmetric degree one and λ-degree zero of the Fedosov-Taylor coefficient $\tau(\phi)$ and the section r (Theorem 1.1), respectively. Note that ϕ_1 is a smooth section in the bundle $T^*M \otimes \Lambda E^*$. Denote by $\Lambda_0 E^*$ the subbundle of the dual Grassmann bundle consisting of elements of even degree. Then ρ is a smooth section in $T^*M \otimes \Lambda_0 E^* \otimes T^*M$. Consider now the bundle $TM \otimes \Lambda_0 E^* \otimes T^*M$. There is an obvious fiberwise associative multiplication $\bullet$ in that bundle which comes from the identification of $TM \otimes T^*M$ with the bundle of linear endomorphism of TM: Let X, Y be vector fields on M, $\phi, \psi \in \Lambda_0 E^*$, and α, β one-forms on M. Then

$$(X \otimes \phi \otimes \alpha) \bullet (Y \otimes \psi \otimes \beta) := (\alpha(Y)) X \otimes (\phi \wedge \psi) \otimes \beta. \tag{37}$$

Let $\hat{R}^E$ be the section in $\Gamma(TM \otimes \Lambda^2 E^* \otimes T^*M)$ whose components in a bundle chart read

$$\hat{R}^E := \frac{1}{4} \Lambda^{ik} R^{(E)}_{ABkj} \partial_i \otimes e^A \wedge e^B \otimes dx^j =: \partial_i \otimes (\hat{R}^E)^i_j \otimes dx^j, \tag{38}$$

and let $\hat{\rho} \in \Gamma(TM \otimes \Lambda_0 E^* \otimes T^*M)$ be defined by

$$\hat{\rho} := \partial_i \otimes \Lambda^{ik} i_s(\partial_k) \rho =: \partial_i \otimes \hat{\rho}^i_j \otimes dx^j. \tag{39}$$

Note that we can form arbitrary power series in $\hat{R}^E$ by using the multiplication $\bullet$ since $\hat{R}^E$ is nilpotent.

We have the following Lemma:

Lemma 1.3 *With the above notations and definitions:*

$$M_1(\phi, \psi) = \Lambda^{ij} (i_s(\partial_i)\phi_1) \wedge (i_s(\partial_j)\psi_1) + q^{AB} (j(e_A)(\phi)) \wedge (i(e_B)(\psi)), \tag{40}$$

$$\phi_1 = dx^j ((1-\hat{\rho})^{-1})^i_j \nabla^E_{\partial_i} \phi, \tag{41}$$

$$\hat{\rho} = 1 - (1 - 2\hat{R}^E)^{1/2}. \tag{42}$$

where $(1-\hat{\rho})^{-1}$ and $(1-2\hat{R}^E)^{1/2}$ denote the corresponding power series with respect to the $\bullet$ multiplication.

Proof. The first equation is a straightforward computation.
For the second, use the Fedosov recursion for $\tau(\phi)$, (Proposition 1.3), note that $\phi_1^{(k)}$ is zero for $k \geq n+2$ and that only the component ρ of r matters since both $\tau(\phi)$ and r depend only on λ^2, sum over the total degree which yields the equation

$$\phi_1 = \delta^{-1} \nabla^E \phi + dx^j (\hat{\rho})^i_j (i_s(\partial_i)\phi_1)$$

which proves the second equation.
For the third, use the Fedosov recursion for r, (Theorem 1.1), take the component

of symmetric degree 1 and λ-degree zero, sum over the total degree, and arrive at the quadratic equation

$$\hat{\rho} - \hat{R}^E = \frac{1}{2}\hat{\rho} \bullet \hat{\rho}.$$

Since r and hence ρ does not contain components of symmetric degree zero, there is only one solution to this equation, namely the above third equation. ■

This Lemma immediately implies the desired formula for the super-Poisson bracket:

Theorem 1.4 *The super-Poisson bracket M_1 obtained by the Fedosov star-product takes the following form:*

$$\begin{aligned} M_1(\phi, \psi) &= \Lambda^{ij}((1-2\hat{R}^E)^{-1/2})_i^k \wedge ((1-2\hat{R}^E)^{-1/2})_j^l \wedge \nabla^E_{\partial_k}\phi \wedge \nabla^E_{\partial_l}\psi \\ &\quad + q^{AB}(j(e_A)(\phi)) \wedge (i(e_B)(\psi)) \end{aligned}$$

Proof. Clear from the Lemma! ■

Corollary 1.1 *The above super-Poisson bracket coincides with the Rothstein super-Poisson bracket $\{\ ,\ \}_R$, see (4) and [31].*

Proof. Since by definition $\Lambda^{ij}(\hat{R}^E)_i^k = \Lambda^{ki}(\hat{R}^E)_i^j$ the same relation holds for any power series (with respect to $\bullet$) $(f(\hat{R}^E))_i^k)$ whence the result. ■

Remarks:

1. In case (M, ω) is Kähler there exist star-products of Wick type on M (see [26, 12]): They are characterized by the property that for any two complex-valued smooth functions f, g on M the star-product $f *' g$ is made out of bidifferential operators which differentiate f in holomorphic directions only and g in antiholomorphic directions only. It seems to me very likely that super analogues of these star-products can readily be formulated for any complex holomorphic Hermitean vector bundle over M as it has been done in geometric quantization, see [22].

2. If the dual Grassmann bundle ΛE^* is replaced by the symmetric power $\vee E^*$ and the fiber metric q by some antisymmetric bilinear form on the fibers covariantly constant by some connection in E the whole construction can presumably carried through as well (see also Neumaier's related construction for differential operators in [10], Section 3). As we shall explain further down this can be interpreted as a particular case of a symplectic fibration.

3. It may also be interesting to compute this construction in the particular case where M is the cotangent bundle of an arbitrary semi-Riemannian manifold Q and E is the tangent bundle of Q pulled back to T^*Q by the bundle projection. Star-products on T^*Q are strongly related to (pseudo) differential operator calculus on Q, see [30, 9, 10, 11]. In that situation one could study asymptotic representation theory incorporating Dirac operators. T. Voronov has studied the algebra $\mathscr{C}$ using symbol calculus and its representations on the space of differential forms on Q (which is an intermediate step towards spinors), see [32].

Note added: The above Fedosov construction is not the full Fedosov construction one would expect in supermanifold theory as I have been made aware by the referee: There the super-Fedosov algebra should rather consist of a sort of completed tensor product of supersymmetric tensor fields (generalizing $\Gamma(\vee T^*M)$) and superdifferential forms (generalizing $\Gamma(\Lambda T^*M)$) which would include our $\mathscr{W} \otimes \Lambda$, but also -roughly speaking- additional symmetric tensors and differential forms 'in the purely fermionic directions'. Moreover the fiberwise multiplication would involve the full Rothstein superbracket. It is very probable that such a super Fedosov construction will go through without any big conceptual problem and, since to my best knowledge this has not yet been done in the literature, will be an interesting problem to attack.

I believe that the rôle of the above Fedosov construction can perhaps best be compared with the constructions which have been done in the meantime by B. Fedosov and O. Kravchenko for ordinary (i.e., non super) symplectic fibrations (see [20, 27]): They are using an *intermediate Fedosov construction* which starts with a 'purely vertical' star-product on the symplectic fibers satisfying some compatibility conditions which is supposed to already exist; in a second step the Fedosov construction proper is then only done for the base, but 'tensored' with the 'vertical algebras': The result is a star-product on the total space. The curvature of the fiber bundle underlying the symplectic fibration enters in the symplectic form of the total space when it is expressed in terms of the symplectic form on the base and on the fibers.

It seems to me that an even symplectic split supermanifold can be regarded as a 'supersymplectic fibration' with symplectic base and 'purely fermionic' fibers, and the simple nature of the Rothstein super symplectic form exactly corresponds to that picture. Moreover, in the Fedosov construction presented in this contribution the 'fermionic vertical direction', viz: The algebra $\Gamma(\Lambda E^*)$ already carries a simple explicit vertical star-product, namely a sort of formal Clifford multiplication (see also the next Section), and the construction is intermediate insofar that symmetric and antisymmetric tensor fields only come from the base. It is an interesting question under which circumstances the 'full' Fedosov construction for even symplectic supermanifolds (which will no doubt be much more complicated) reduces to the above 'intermediate construction'.

2. Flat vector bundles

An important particular case is given by a vector bundle E with fiber metric q on which there exists a *flat* covariant derivative ∇^E, for instance in the case of a trivial bundle $M \times \mathbb{R}^n$ with q being a nondegenerate bilinear form on $\mathbb{R}^n$ not depending on M.

Note first the standard fact for flat vector bundles that there is an open cover $(U_\alpha)_{\alpha \in I}$ of M together with a basis of local sections e_A, $1 \leq A \leq n$ defined on each U_α which are covariantly constant and which are related by constant transition matrices on the overlaps of any two of the U_α.

We have the following Lemma:

Lemma 2.1 *With the above additional assumptions the following holds:*

1. *The map r as defined in Theorem 1.1 does not depend on ΛE^*, i.e., is contained in $\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M \otimes \Lambda^1 T^*M))[[\lambda]]$.*
2. *The Fedosov-Taylor series of a function $f \in C^\infty(M)$ does not depend on ΛE^*, i.e., is contained in $\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M))[[\lambda]]$.*
3. *The Fedosov-Taylor series of a local covariantly constant section ϕ in $\Gamma(\Lambda E^*)$ is equal to ϕ.*

Proof. 1. Since $r^{(3)} = R^{(M)}$, and since δ^{-1} and ∇ preserve $\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M \otimes \Lambda T^*M))[[\lambda]]$ which is a fiberwise subalgebra of $\mathscr{W} \otimes \Lambda$ the statement follows by induction using Theorem 1.1.
2. The proof is completely analogous to part 1. upon using the formulas in Proposition 1.3.
3. Again by induction using Proposition 1.3 where the fact is used that r supercommutes with $\Gamma(\Lambda E^*)$ according to 1. ■

This immediately implies the following formula for the star-product:

Theorem 2.1 *We make the above assumptions. Let ϕ, ψ two sections in $\mathscr{C}$ and express them locally as $\phi = \sum_{d=0}^{n} \frac{1}{d!}\phi_{A_1\cdots A_d} e^{A_1} \wedge \cdots \wedge e^{A_d}$ and likewise for ψ where e^A, $1 \leq A \leq n$, is a local basis of covariantly constant sections of E^* and $\phi_{A_1\cdots A_d}$ are local C^∞-functions. Then*

$$\phi * \psi = \sum_{d,d'=0}^{n} \frac{1}{d!}\frac{1}{d'!}(\phi_{A_1\cdots A_d} *_{\mathrm{F}} \psi_{B_1\cdots B_{d'}})(e^{A_1} \wedge \cdots \wedge e^{A_d} *_{\mathrm{Cl}} e^{B_1} \wedge \cdots \wedge e^{B_{d'}}) \quad (43)$$

where $*_{\mathrm{F}}$ denotes the usual Fedosov star-product on M defined by the map r (restricted to $\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M \otimes \Lambda^1 T^*M))[[\lambda]]$) and $*_{\mathrm{Cl}}$ denotes the formal tensorial Clifford multiplication in $\mathscr{C}$ defined by

$$\phi *_{\mathrm{Cl}} \psi := \sum_{l=0}^{n} \frac{(\mathrm{i}\lambda/2)^l}{l!} q^{A_1B_1} \cdots q^{A_lB_l} \big(j(e_{A_1}) \cdots j(e_{A_l})\phi\big) \wedge \big(i(e_{B_1}) \cdots i(e_{B_l})\psi\big).$$

Formula (43) does not depend on the chosen covariantly constant local trivializaton.

Proof. The subalgebra $\times_{s=0}^{\infty}\Gamma(\mathbb{C}(\vee^s T^*M \otimes \Lambda T^*M))[[\lambda]]$ of $\mathscr{W}$ is clearly preserved by the Fedosov derivative D whence it follows at once that $f * g = f *_{\mathrm{F}} g$ for all $f, g \in C^\infty(M)[[\lambda]]$. Moreover, for a covariantly constant section χ of $\mathscr{C}$ we clearly have $f\chi = \sigma(\tau(f)\tau(\chi)) = \sigma(\tau(f) \circ \tau(\chi))$ using the above Lemma and the properties of $\circ$ whence $f\chi = f * \chi = \chi * f$. Finally, note that $\chi * \chi' = \chi \circ \chi' = \chi *_{\mathrm{Cl}} \chi'$ for two covariantly constant sections, where the result is again covariantly constant, and therefore

$$(f\chi) * (g\chi') = f * \chi * g * \chi' = f * g * \chi * \chi' = (f *_{\mathrm{F}} g)(\chi *_{\mathrm{Cl}} \chi'),$$

which proves the above formula. Since the transition functions are constant it follows that (43) does not depend on the chosen local basis of covariantly constant sections. ■

Conversely, it is easy to see that the above formula (43) always defines an associative $\mathbb{Z}_2$-graded deformation of $\mathscr{C}_0$ where $*_{\mathrm{F}}$ can be replaced by any given star-product on M: It is locally given by the tensor product over $\mathbb{C}[[\lambda]]$ of the associative algebra $(C^\infty(M)[[\lambda]]$ with the formal Clifford algebra $(\Lambda\mathbb{R}^n[[\lambda]], *_{\mathrm{Cl}})$.

For a trivial flat bundle without holonomy the above formula had been given by R. Eckel in his thesis [16], p. 66.

3. A quantum BRST complex for quantum covariant star-products

The results of this Section have been obtained in collaboration with Hans-Christian Herbig and Stefan Waldmann in [8].

Let (M, ω) a symplectic manifold. Suppose that a Lie group G (with Lie algebra $\mathfrak{g}$) symplectically and properly acts on M (e.g., when G is compact) allowing for a classical momentum map $J : M \to \mathfrak{g}^*$: For each $\xi \in \mathfrak{g}$ let ξ_M be the fundamental field $m \mapsto d/dt(exp(t\xi)m)|_{t=0}$, then $\omega^\flat(\xi_M) = d\langle J, \xi\rangle$ and $J(gm) = \mathrm{Ad}^*(g)J(m)$ for all $g \in G, m \in M$. This implies the Lie homomorphism property

$$\{\langle J, \xi\rangle, \langle J, \eta\rangle\} = \langle J, [\xi, \eta]\rangle, \tag{44}$$

for all $\xi, \eta \in \mathfrak{g}$. Recall the Marsden-Weinstein phase space reduction scheme, [28]: Suppose for the rest of this Section that 0 is a regular value of J and that the constraint surface $C := J^{-1}(0)$ is nonempty. Then G acts locally freely on C, and supposing that G acts freely and properly on C the quotient manifold $M_{\mathrm{red}} := C/G$ becomes a symplectic manifold, its symplectic form ω_{red} being determined by the condition that its pull-back to C by the canonical projection equals the restriction of ω to TC. Note that each G-invariant smooth function on C naturally projects to M_{red}.

Now let $*$ be a star-product on M. According to [2] and [33] a formal power series $\boldsymbol{J} = \sum_{r=0}^{\infty} \lambda^r \boldsymbol{J}_r \in C^\infty(M, \mathfrak{g}^*)[[\lambda]]$ will be called a quantum momentum map and the star-product $*$ (quantum) covariant iff $\boldsymbol{J}_0 = J$ and analogously to (44):

$$\langle \boldsymbol{J}, \xi \rangle * \langle \boldsymbol{J}, \eta \rangle - \langle \boldsymbol{J}, \eta \rangle * \langle \boldsymbol{J}, \xi \rangle = \mathbf{i}\lambda \langle \boldsymbol{J}, [\xi, \eta] \rangle, \tag{45}$$

for all $\xi, \eta \in \mathfrak{g}$. We call $(M, *, G, \boldsymbol{J}, C)$ satisfying the previous conditions a *Hamiltonian quantum G-space with regular constraint surface*. According to a Theorem by Fedosov [19, Sect. 5.8] an even stronger condition can be achieved for all such group actions preserving a connection, e.g., proper actions (since they always preserve a Riemannian metric), namely strong invariance:

$$\langle J, \xi \rangle * f - f * \langle J, \xi \rangle = \mathbf{i}\lambda \{ \langle J, \xi \rangle, f \}, \tag{46}$$

which obviously implies (45) setting $\boldsymbol{J} = J$. A simple example is provided by the standard Moyal-Weyl star-product on $\mathbb{R}^{2n}$ together with the Lie algebra of all infinitesimal linear symplectic transformations represented by the space of all quadratic homogeneous polynomials. The problem whether a general classical momentum map can be deformed into a quantum momentum map for a suitable star-product is still an open problem as far as I know.

We are now constructing a BRST complex related to that problem (see for a general introduction the book [24] and our article [8] for more references): Consider the trivial bundle $E := (\mathfrak{g} \oplus \mathfrak{g}^*) \times M$ together with the fiber metric q defined by the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Then the superobservable algebra $\mathscr{C}$ (called $\mathscr{A}$) in [8]) of the first Section equals

$$\mathscr{C} = \Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \otimes C^\infty(M)[[\lambda]]. \tag{47}$$

As a $\mathbb{C}[[\lambda]]$-module this space carries natural $\mathbb{Z}$-gradings, namely the *ghost degree* (form degree in $\Lambda\mathfrak{g}^*$), the *antighost degree* (form degree in $\Lambda\mathfrak{g}$), and the *ghost number* Gh which is defined as the difference of the ghost degree and the antighost degree and which we shall consider as a $\mathbb{C}[[\lambda]]$-linear map $\mathscr{C} \to \mathscr{C}$ with the ghost number integers as eigenvalues. We shall write $\mathscr{C}^{i,j}$ for the submodule of all those elements having ghost degree i and antighost degree j and $\mathscr{C}^{(i)}$ for the submodule of all those elements having ghost number i. We equip $\mathscr{C}$ with a star-product as in Section 2, (43) where the initial star-product on M does not have to be of Fedosov type. Consider now the following three elements of $\mathscr{C}$: $\boldsymbol{J} \in \mathscr{C}^{1,0}$, $\Omega := -1/2[\ ,\ ] \in \mathscr{C}^{2,1}$, and $\gamma :=$ one half of the identity homomorphism of $\mathfrak{g}$, contained in $\mathscr{C}^{1,1}$. Let $\Theta := \boldsymbol{J} + \Omega$, the so-called BRST-charge which is contained in $\mathscr{C}^{(1)}$. Define the *BRST operator Q* by

$$Q(\phi) := \frac{1}{\mathbf{i}\lambda} \left(\Theta * \phi - (-1)^{a+b} \phi * \Theta \right), \quad \forall a, b \in \mathbb{Z}, \forall \phi \in \mathscr{C}^{a,b}. \tag{48}$$

Then we have the following Theorem:

Theorem 3.1 *Let $(M,*,G,J,C)$ be a Hamiltonian quantum G-space with regular constraint surface. Then*

1. *The Ghost number operator* Gh *is equal to $\phi \mapsto \frac{1}{\mathrm{i}\lambda}(\gamma * \phi - \phi * \gamma)$ and therefore is a derivation of $(\mathscr{C},*)$ which thus becomes a $\mathbb{Z}$-graded associative algebra.*
2. $\Theta * \Theta = 0$.
3. *The BRST operator has square zero, $Q^2 = 0$, and is a superderivation of ghost number one of $(\mathscr{C},*)$.*

The proof of this statement is a rather straightforward consequence of Equation (45). For more details see [8].

Define the *quantum BRST cohomology* by $\mathrm{Ker}Q/\mathrm{Im}Q =: \boldsymbol{H}_{\mathrm{BRST}}(\mathscr{C}[[\lambda]])$. Then we have the following

Theorem 3.2 *Let $(M,*,G,J,C)$ be a Hamiltonian quantum G-space with regular constraint surface. Then*

1. *$\boldsymbol{H}_{\mathrm{BRST}}(\mathscr{C}[[\lambda]])$ becomes a $\mathbb{Z}$-graded associative algebra in a canonical way.*
2. *There is a representation $\boldsymbol{\varrho}_C$ of the Lie algebra $\mathfrak{g}$ on the $\mathbb{C}[[\lambda]]$-module $C^\infty(C)[[\lambda]]$ deforming the representation ϱ_C induced by the restriction of the fundamental fields to C such that the quantum BRST cohomology is isomorphic to the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with values in $C^\infty(C)[[\lambda]]$ with respect to $\boldsymbol{\varrho}_C$.*
3. *In particular, the component of ghost number zero of the quantum BRST cohomology is isomorphic to the submodule of all those elements in $C^\infty(C)[[\lambda]]$ which are invariant under $\boldsymbol{\varrho}_C$.*

See again [8] for a detailed proof.

In case the Hamiltonian action of the connected Lie group G on M is proper and the reduced phase space exists, we can choose a strongly invariant star-product on M (see (46)). Under these circumstances we have the stronger Theorem:

Theorem 3.3 *With the assumption of the previous Theorem and the above additional assumptions we have:*

1. *The quantum BRST-cohomology is isomorphic to the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with values in $C^\infty(C)[[\lambda]]$ with respect to the undeformed representation ϱ_C.*
2. *In particular, the component of ghost number zero of the quantum BRST cohomology is isomorphic to the submodule of all those elements in $C^\infty(C)[[\lambda]]$ which are invariant under ϱ_C. This space being isomorphic to $C^\infty(M_{\mathrm{red}})[[\lambda]]$ the algebra structure on the cohomology induces a star-product on the reduced space M_{red}.*

For a proof see [8].
Remarks:

1. The proofs of the last two theorems are rather technical. They heavily rely on one side on purely geometric considerations, namely the existence of tubular neighborhoods (which can be chosen G-invariant for proper G-actions) and the triviality of the normal bundle of C in M (since 0 is a regular value of J), which leads to the construction of an acyclic Koszul complex (first on the submodule of $\mathscr{C}$ of ghost degree zero which is in a standard way extended to all of $\mathscr{C}$), and a rather explicit chain homotopy for that complex analogous to the one used in the proof of Poincaré's Lemma. Secondly, we have used a purely tensorial, explicit equivalence transformation which modifies the Clifford part of the multiplication in $\mathscr{C}$ in such a way that Q splits into a boundary operator lowering the antighost degree by 1 and leaving invariant the ghost degree (which turns out to be a deformation of the aforementioned Koszul boundary operator) and a coboundary operator raising the ghost degree by one and leaving invariant the antighost degree (which turns out to be equal to a certain Chevalley-Eilenberg operator of $\mathfrak{g}$). Hence $\mathscr{C}$ becomes a double complex where one differential is acyclic. This fact has been known in the classical situation, but miraculously remains true in this deformed situation. Thirdly, to relate the total cohomology to the data on the constraint surface C we use an augmentation of this complex consisting in a *deformation of the restriction map* by a formal series of differential operators which can be constructed out of Q and the classical chain homotopies. Finally, in the case of a proper group action the resulting star-product on the reduced space can be related to the one on M essentially by means of the deformed restriction map.

2. For quantum covariant, but *not* strongly invariant star-products it can happen that the above mentioned ghost number zero part of the cohomology, the space of 'quantum G-invariant functions on C', can be too small in the sense that it is no longer a deformation of the whole space of classical G-invariant functions, but of a subspace of the latter, which is quite an anomaly. In a simple example (see [8], Section 7) we have seen that the reduced algebra can ultimately become commutative which does no longer seem to resemble a reasonable reduction of quantization, but which — with a little bad luck — in principle is possible as the example shows.

4. Classical reducible BRST without ghosts of ghosts

The results of this Section have been obtained in collaboration with Hans-Christian Herbig in [7].

Let C be an arbitrary closed coisotropic submanifold of a symplectic manifold (M,ω) of codimension n, i.e., the ω-orthogonal space to each tangent space of C is contained in that tangent space. Physicists would speak of C as a 'first class constraint surface'. Let TC^{ω} be the ω-orthogonal bundle to TC. This is known to be an integrable subbundle of TC and gives rise to a local foliation thanks to Frobenius' Theorem. If this foliation allows for a smooth quotient manifold M_{red} it becomes a symplectic manifold in a canonical way, see, e.g., [1], p. 417–418. Fix a subbundle N of $TM|_C$ such that $TM|_C = N \oplus TC$ (e.g., as the normal bundle to TC with respect to some Riemannian metric). The symplectic form provides an identification of N with the dual of TC^{ω} via $v \mapsto (w \mapsto \omega(v,w))$ where $c \in C, v \in N_c$ (the fiber of N over c) and $w \in T_cC^{\omega}$ and an identification of TC^{ω} with the conormal bundle of TC, i.e., the subbundle TC^{ann} of $T^*M|_C$ of all those cotangent vectors annihilating TC via $v \mapsto (w \mapsto \omega(v,w))$ where $c \in C$, $v \in T_cC^{\omega}$ and $w \in T_cM$, whence

$$N^* \cong TC^{\omega} \cong TC^{\mathrm{ann}}. \tag{49}$$

The nontriviality of the bundle N (and hence of the two others in the above equation) is related to the physicists' 'reducible case': Here the submanifold C is given as the zero locus of a finite set of in general not functionally independent smooth real valued functions.

Next, choose a tubular neighborhood around C, i.e., an open neighborhood U of the zero-section of N together with a diffeomorphism Φ of U onto an open neighborhood V of C in M such that $\Phi(c) = c$ for all $c \in C$ (where we identify C with the zero-section in N). Hence U becomes a symplectic manifold with the pulled-back form $\Phi^*\omega$. Denoting the bundle projection $N \to C$ by p we consider the pulled-back bundle p^*N over U. We shall denote the dual bundle of p^*N by F, whence p^*N can be identified with F^*. We have made this choice of notation to have an analogy $M \times \mathfrak{g} \cong F$ and $M \times \mathfrak{g}^* \cong F^*$ with the previous section.

The main idea which will make the construction work is the fact that the bundle $F^*(= p^*N)$ admits the tautological section J which maps each point u of U to the same point in the fiber over $p(u)$. J can be seen as a generalization of the momentum map of the previous section. I had been inspired by a similar construction in Connes's book [13], p. 210, used for the computation of the Hochschild cohomology of the algebra of all complex-valued C^{∞}-functions on a given manifold M.

Choosing an arbitrary covariant derivative ∇^F in F (inducing a covariant derivative ∇^{F^*} in F^* in the standard way) we set

$$E := F \oplus F^*; \quad \nabla^E := \nabla^F + \nabla^{F^*} \tag{50}$$

and choose the natural pairing between F and F^* as fiber metric q. It is clear that the above ∇^E preserves q.

Consider now $\mathscr{C}_0 := \Gamma(\Lambda F^* \otimes \Lambda F)$ together with the Rothstein super-Poisson bracket $\{\ ,\ \}_R$ constructed out of the above data. Define the ghost degree, antighost degree, and ghost number maps in the same way as in the previous section. Then we have the following

Theorem 4.1 *We use the above-made assumptions. Then*

1. *The ghost number map* Gh *is a derivation of the super-Poisson algebra* $\mathscr{C}_0$ *which thus becomes* $\mathbb{Z}$*-graded.*

2. *There is an element* $\Theta := \sum_{i=0}^{n} \Theta_i \in \mathscr{C}_0$*, the so-called classical BRST charge, such that* $\mathsf{Gh}(\Theta) = 1$*,* $\Theta_0 = J$*, the antighost degree of* Θ_i *is i, and, most importantly,* $\{\Theta, \Theta\}_R = 0$*.*

3. *The classical BRST operator* $Q := \{\Theta, \ \}_R$ *has square zero, increases the ghost-number by one, and its classical BRST-cohomology* $\mathrm{Ker}Q/\mathrm{Im}Q$ *carries a canonical* $\mathbb{Z}$*-graded super-Poisson algebra structure induced by the one on* $\mathscr{C}_0$*.*

In order to compute the above cohomology we consider the space of *vertical differential forms on C*, i.e., the space of sections $\Omega_v(C) := \Gamma(\Lambda(TC^\omega)^*)$ together with the vertical exterior derivative d_v which is defined by the same formula as the standard exterior derivative but restricted to vertical vector fields, i.e., sections of the integrable subbundle TC^ω. Then we have the following result (which should be known by other methods):

Theorem 4.2 *We use the above-made assumptions and notations. Then the classical BRST-cohomology is isomorphic to the vertical de Rahm cohomology, i.e., the cohomology of the complex* $(\Omega_v(C), d_v)$*. This latter space thus carries the structure of a* $\mathbb{Z}$*-graded super-Poisson bracket. Moreover, the sector of the classical BRST-cohomology having vanishing ghost number exactly corresponds to the space of all complex-valued* C^∞*-functions on C which are constant on the connected leaves of the foliation defined by* TC^ω*. In case the reduced space* M_{red} *exists this last space is equal to the space of all complex-valued* C^∞*-functions on* M_{red}*.*

For details of the proof, see [7]. The main tool is the fact that J defines a Koszul boundary operator on the space $\Gamma(\Lambda F)$ in the same way as has been remarked in the previous Section, that the resulting complex is acyclic allowing for an augmentation map consisting of the restriction to C, and that the component of $\{J, J\}_R$ having vanishing antighost degree vanishes when restricted to C thanks to the fact that C is coisotropic and to the chosen connection ∇^E. In the irreducible case where C is given as the zero locus of $n := codimC$ functionally independent functions the method of deforming J is well known, see, e.g., [24]

The advantage of the above construction is that it is contained in a simple, geometrically defined BRST-complex $\mathscr{C}_0$ with only a finite number of nonzero $\mathscr{C}^{i,j}$ (although it may become difficult to explicitly compute the tubular neighborhoods) in contrast to the more elaborate multi-step ghosts-of-ghosts methods based on spectral sequence techniques, [21]. It is tempting to try a quantization of this complex by means of the Fedosov-type star-product constructed in the first section, but this would require a more sophisticated analysis of the (affine) geometry of the vicinity of C to solve the obvious problem whether the component of antighost degree zero of $J * J$ vanishes when restricted to C.

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On positive deformations of $*$-algebras[‡]

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Abstract. Motivated by deformation quantization we consider $*$-algebras over ordered rings and their deformations: we investigate formal associative deformations compatible with the $*$-involution and discuss a cohomological description in terms of a Hermitian Hochschild cohomology. As an ordered ring allows for a meaningful definition of positive functionals and as the formal power series with coefficients in an ordered ring are again an ordered ring we define a deformation to be positive if any positive linear functional of the undeformed algebra can be deformed into a positive linear functional of the deformed algebra. We discuss various examples and prove in particular that star-products on symplectic manifolds are positive deformations.

Keywords: positive deformations, deformation quantization, $*$-algebras, ordered rings

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Dedicated to the memory of Moshé Flato

1. Introduction

In this paper we discuss several aspects of deformations of algebras with a $*$-involution which are defined over an ordered ring. Our main example is deformation quantization as introduced by Bayen, Flato, Frønsdal, Lichnerowicz, and Sternheimer in [3], see e.g. [41, 39] for recent surveys. In deformation quantization the algebra of classical observables, the smooth complex-valued functions $C^\infty(M)$ on a Poisson manifold, is deformed into the quantum mechanical observable algebra by introducing a new, $\hbar$-dependent associative product, the star product, such that in zeroth order of $\hbar$ the star-product coincides with the pointwise product and the commutator yields in first order i times the Poisson bracket. In deformation quantization the star-product usually is regarded as formal, i.e., based on formal power series in $\hbar$, and thus a star-product is a formal associative deformation in the sense of Gerstenhaber [26]. The existence of star-products was shown by De Wilde and Lecomte [22], Fedosov [23, 24], and Omori, Maeda, and Yoshioka [35] for the case of symplectic manifolds, and by Kontsevich [30] for the general case of a

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Poisson manifold. The classification up to equivalence is due to Nest and Tsygan [32, 33], Bertelson, Cahen, and Gutt [4], Deligne [21], Weinstein and Xu [42], and Kontsevich [30]. On the other hand the formal character of these star products is certainly not sufficient from the physical point of view whence one also has to study convergence properties with respect to $\hbar$. This can be done either by starting with a formal star product and then finding suitable subalgebras where one can make sense out of convergence, see e.g. [16, 17, 18, 19, 5, 8, 9, 7] and references therein, or directly by formulating the deformation within the framework of e.g. C^*-algebras from the beginning, which leads e.g. to Rieffel's notion of strict deformation quantization, see [38] and Landsman's book [31] for related approaches and references. In many cases the formal approach can be seen as an 'asymptotic expansion' in $\hbar$ of a convergent situation, which can be made more precise in several examples as e.g. for cotangent bundles [8, 9, 7], and it is very useful to think of formal deformations in that way even if it is not clear if there exists a corresponding strict deformation (see also [21, 36] for a more sheaf theoretic approach). This leads to the following program, namely trying to find 'formal analogs' of various techniques known in C^*-algebra theory in such a way that they can be viewed as 'asymptotic expansions' and thus to understand the classical and semi-classical limit of these constructions which possibly opens also an asymptotic point of view to Connes' non-commutative geometry [20]. This approach has turned out to be interesting both from the physical and mathematical point of view: in [14, 13] Bordemann and Waldmann have introduced a notion of formal GNS representations for star-products on pre-Hilbert spaces over the ring of formal power series by noticing that the ring of real formal power series is in fact ordered and thus allows to define positive functionals etc. as in the C^*-algebra theory. This approach (re-)produces many physically interesting results for the star-product algebras in a purely formal framework, as e.g. for cotangent bundles the Schrödinger-like differential operator representations, Aharonov-Bohm-like effects and representations on sections of line bundles when magnetic monopoles are present [8, 9, 7]. Also thermodynamical KMS states and their representations have been investigated, see [1, 2] as well as [10, 11] and [40] for a related 'formal' Tomita-Takesaki theory. On the other hand the framework of $*$-algebras and pre-Hilbert spaces over ordered rings seems to be mathematically interesting and turns out to have a rich structure. In [15] we started to develop the notions of algebraic Rieffel induction and formal Morita equivalence for such $*$-algebras in order to understand the (semi-)classical limits of the corresponding constructions for C^*-algebras, as e.g. used by Landsman [31] to describe quantum analogous of phase space reduction. As this also has been done in the framework of deformation quantization, see e.g. [25, 6], it would be very interesting to compare these approaches. During our investigations we needed the notion of a *positive deformation* of a $*$-algebra in the sense that all positive linear functionals can be deformed into positive linear functionals of the deformed algebra. This is in some sense 'dual' to Landsman's notion of a positive quantization map, but less restrictive, and it seems to be more suitable for our purposes.

In this paper we shall discuss the deformations of $*$-algebras over ordered rings in general and apply our results to the main examples of deformation quantization obtaining the following results: after reviewing briefly the notions of ordered rings and $*$-algebras and discussing the notion of 'sufficiently many positive linear functionals' in Section 2, we set up the deformation program for $*$-algebras in Section 3 where the 'real part' of the Hochschild cohomology is relevant for Hermitian deformations, i.e., those deformations which do not deform the $*$-involution. In Section 4 we motivate the notion of a positive deformation and discuss some basic constructions proving in particular that positive deformations of $*$-algebras with sufficiently many positive linear functionals still have sufficiently many positive linear functionals. Finally, in Section 5 we prove that star-products on symplectic manifolds are positive deformations and that on arbitrary Poisson manifolds star-products have sufficiently many positive linear functionals.

2. $*$-algebras over ordered rings

In this section we shall recall some basic definitions and examples concerning ordered rings and $*$-algebras. For further references and details see e.g. [14, 15]. Let R be an *ordered ring*, i.e., an associative, commutative and unital ring with a subset $\mathsf{P} \subset \mathsf{R}$ such that R is the disjoint union $\mathsf{R} = -\mathsf{P} \cup \{0\} \cup \mathsf{P}$ and $\mathsf{P} \cdot \mathsf{P} \subseteq \mathsf{P}$, $\mathsf{P} + \mathsf{P} \subseteq \mathsf{P}$. The elements in P are called positive. Clearly R has characteristic zero and no zero divisors. Then consider the quadratic ring extension $\mathsf{C} = \mathsf{R}(\mathrm{i})$ where $\mathrm{i}^2 := -1$. Complex conjugation $z = a + \mathrm{i}b \mapsto \bar{z} := a - \mathrm{i}b$ with $a, b \in \mathsf{R}$ is defined as usual and clearly $z \in \mathsf{R} \subset \mathsf{C}$ iff $z = \bar{z}$ and $\bar{z}z \geq 0$.

Let $\mathscr{A}$ be an associative algebra over C endowed with a $*$-involution, i.e., an involutive C-antilinear anti-automorphism $^* : \mathscr{A} \to \mathscr{A}$, then $\mathscr{A}$ is called a *$*$-algebra* over C. As usual we define *Hermitian*, *unitary* and *normal elements* in C. Given such a $*$-algebra we define a linear functional $\omega : \mathscr{A} \to \mathsf{C}$ to be *positive* if $\omega(A^*A) \geq 0$ for all $A \in \mathscr{A}$. If $\mathscr{A}$ is unital then a positive linear functional is called a state if $\omega(\mathbf{1}) = 1$. For any positive linear functional one has the Cauchy-Schwarz inequality $\omega(A^*B)\overline{\omega(A^*B)} \leq \omega(A^*A)\omega(B^*B)$ and $\omega(A^*B) = \overline{\omega(B^*A)}$ for $A, B \in \mathscr{A}$. Next we consider positive algebra elements in $\mathscr{A}$. There are essentially two ways to define positivity [15, Def. 2.3]: firstly, we call $A \in \mathscr{A}$ *algebraically positive* if $A = a_1 A_1^* A_1 + \cdots + a_n A_n^* A_n$, where $n \in \mathbb{N}$, $a_i \geq 0$, and $A_i \in \mathscr{A}$ for $i = 1, \ldots, n$, and we denote the set of all algebraically positive elements by $\mathscr{A}^{++}$. Secondly, we define a Hermitian element A to be *positive* if for all positive linear functionals $\omega : \mathscr{A} \to \mathsf{C}$ one has $\omega(A) \geq 0$, and we denote the positive elements by $\mathscr{A}^+$. Then clearly $\mathscr{A}^{++} \subseteq \mathscr{A}^+$ and both sets are convex cones with the property that for any $C \in \mathscr{A}$ one has $C^* \mathscr{A}^{++} C \subseteq \mathscr{A}^{++}$ as well as $C^* \mathscr{A}^+ C \subseteq \mathscr{A}^+$.

The basic examples of such $*$-algebras are given by C itself, in which case $\mathsf{C}^+ = \mathsf{C}^{++}$ coincides with the non-negative elements in R. More generally, one can consider $M_n(\mathsf{C})$, the $n \times n$ matrices over C with the usual $*$-structure. As in the case of $\mathsf{C} = \mathbb{C}$ one can show that the positive linear functionals can be written as $A \mapsto \mathsf{tr}(\rho A)$ where tr is the trace functional and ρ is a Hermitian matrix such that $\langle v, \rho v\rangle \geq 0$ for all $v \in \mathsf{C}^n$, where $\langle \cdot, \cdot \rangle$ denotes the usual Hermitian product on C^n. Moreover, $A = A^* \in M_n(\mathsf{C})^+$ if and only if $\langle v, Av\rangle \geq 0$ for all $v \in \mathsf{C}^n$. In the case of deformation quantization we consider the space of complex-valued smooth functions $C_0^\infty(M)$ with compact support (only for technical reasons) on a Poisson manifold as the space of classical observables with the obvious $*$-algebra structure. Then the positive linear functionals are positive Borel measures with finite volume for every compact set and $C_0^\infty(M)^+$ consists of functions with $f(x) \geq 0$ for all $x \in M$ as expected. Nevertheless we should also mention that the above definitions lead to a somehow 'pathological' characterization of positive elements in the case of polynomial algebras, see e.g. [37]. The final example we would like to mention is the Grassmann algebra $\bigwedge(\mathsf{C}^n)$, where we define a $*$-involution by requiring $\mathbf{1}^* = \mathbf{1}$ and $e_i^* = e_i$, where $e_1, \ldots, e_n$ is the canonical basis of C^n. Then one easily shows that $\omega : \bigwedge(\mathsf{C}^n) \to \mathsf{C}$ is positive if and only if $\omega(\mathbf{1}) \geq 0$ and $\omega(e_{i_1} \wedge \cdots \wedge e_{i_r}) = 0$ for all $r \geq 1$ and $i_1, \ldots, i_r$. Thus in this example one has (up to normalization) only one positive linear functional.

To avoid such examples we state the following definition [15, Def. 2.7]: A $*$-algebra $\mathscr{A}$ has sufficiently many positive linear functionals if for every Hermitian element $H \neq 0$ there exists a positive linear functional ω such that $\omega(H) \neq 0$. Clearly $M_n(\mathsf{C})$ and $C_0^\infty(M)$ have sufficiently many positive linear functionals but $\bigwedge(\mathsf{C}^n)$ does not. This condition turns out to be rather strong as it implies already some very C^*-algebra like properties: one can show that if $\mathscr{A}$ has sufficiently many positive linear functionals then, e.g. $A^*A = 0$ implies $A = 0$ as well as $H^n = 0$ for a normal element H implies $H = 0$. Moreover, there exists a faithful $*$-representation on a pre-Hilbert space over C which can be constructed by means of the GNS construction, see [15, Prop. 2.8].

3. Deformations of $*$-algebras

Before studying deformations of $*$-algebras over ordered rings one notices that if R is an ordered ring then the ring of formal power series $\mathsf{R}[[\lambda]]$ is again ordered in a canonical way, namely one defines $a = \sum_{r=r_0}^\infty \lambda^r a_r$ to be positive if $a_{r_0} > 0$. Thus the deformation program stays in the same category of $*$-algebras over ordered rings (and their quadratic ring extension by 'i'). In the case of deformation quantization the deformation parameter λ plays the role of $\hbar$ and may be substituted $\lambda \mapsto \hbar$ in *convergent* situations.

Denote by $\mu_0 : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ the C-bilinear associative product of $\mathscr{A}$ and by $I_0 : \mathscr{A} \to \mathscr{A}$ the $*$-involution, which we shall frequently refer to as the 'classical' structures. Then a *formal $*$-algebra deformation* of $\mathscr{A}$ is a formal deformation $\mu = \mu_0 + \sum_{r=1}^\infty \lambda^r \mu_r$ of μ_0 to an associative $\mathsf{C}[[\lambda]]$-bilinear product μ for $\mathscr{A}[[\lambda]]$ together with a formal deformation $I = I_0 + \sum_{r=1}^\infty \lambda^r I_r$ of the $*$-involution I_0 to a

$\mathsf{C}[[\lambda]]$-antilinear involutive anti-automorphism of μ. Two such deformations (μ, I) and $(\tilde{\mu}, \tilde{I})$ are called *equivalent* if there exists a formal series $T = \mathsf{id} + \sum_{r=1}^{\infty} \lambda^r T_r$ of C-linear maps $T_r : \mathscr{A} \to \mathscr{A}$ such that $T(\mu(A \otimes B)) = \tilde{\mu}(TA \otimes TB)$ and $TI = \tilde{I}T$. Due to the physical interpretation in deformation quantization of the Hermitian elements with respect to the classical $*$-involution as observables (which are the real-valued functions), we are mainly interested in those $*$-algebra deformations where the classical $*$-involution is *not* deformed, since the classical observables should be observable in the 'quantized' theory, too. This particular kind of $*$-algebra deformations shall be called *Hermitian deformations* of $\mathscr{A}$ (or symmetric, see [3] for this definition in the context of star products). Equivalence of Hermitian deformations now simply means that there is a real equivalence transformation.

We shall now briefly sketch the deformation theory for Hermitian deformations in terms of a Hermitian Hochschild cohomology for $\mathscr{A}$, see [26] for the general cohomological approach to deformations of algebras. On the Hochschild complex $C^\bullet(\mathscr{A}) = \bigoplus_{n=0}^{\infty} \mathsf{Hom}(\mathscr{A}^{\otimes n}, \mathscr{A})$ one defines an action of the $*$-involution (we sometimes use again the symbol $*$ instead of I_0) by

$$\varphi^*(A_1 \otimes \cdots \otimes A_n) := (\varphi(A_n^* \otimes \cdots \otimes A_1^*))^*, \tag{3.1}$$

where $A_1, \ldots, A_n \in \mathscr{A}$ and φ is a n-cochain. Clearly φ^* is again an n-cochain and the map $\varphi \mapsto \varphi^*$ is C-antilinear and involutive. Note that we have also reverted the order of the arguments. Then one finds by a straightforward computation the following relation for the Gerstenhaber product $\varphi \diamond \psi$ of two cochains $\varphi \in C^n(\mathscr{A})$, $\psi \in C^m(\mathscr{A})$:

$$(\varphi \diamond \psi)^* = (-1)^{(n-1)(m-1)} \, \varphi^* \diamond \psi^*. \tag{3.2}$$

The fact that $*$ is an anti-automorphism of μ_0 is expressed by the relation

$$\mu_0^* = \mu_0, \tag{3.3}$$

whence we obtain for the Hochschild differential with respect to μ_0 for an n-cochain φ

$$(\delta\varphi)^* = (-1)^{n-1} \delta \varphi^*, \tag{3.4}$$

where δ is the Gerstenhaber bracket with μ_0, i.e., the graded $\diamond$-commutator $\delta\varphi = (-1)^{n-1}[\mu_0, \varphi]$. We call a cochain φ *Hermitian* if $\varphi^* = \varphi$ and anti-Hermitian if $\varphi^* = -\varphi$, and denote the *Hermitian cochains* by $C_H^\bullet(\mathscr{A})$. They are a $\mathbb{Z}$-graded R-submodule of $C^\bullet(\mathscr{A})$. For a 0-cochain this coincides with the notion of a Hermitian algebra element while a 1-cochain is Hermitian if $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathscr{A}$. Moreover, for $\varphi \in C_H^n(\mathscr{A})$ the cochain $\delta\varphi$ is Hermitian if n is odd and anti-Hermitian if n is even. This observation leads to the following definition of the Hermitian Hochschild cohomology of $\mathscr{A}$. Define $Z_H^n(\mathscr{A})$ to be the set of Hermitian cocycles and define $B_H^n(\mathscr{A})$ to be the set of Hermitian coboundaries $\varphi = \delta\psi$ where ψ is Hermitian if n even and anti-Hermitian if n is odd. Then

$H^n_H(\mathscr{A}) := Z^n_H(\mathscr{A})/B^n_H(\mathscr{A})$ denotes the n-th *Hermitian Hochschild cohomology group*. As an ordered ring R has characteristic zero and no zero-divisors it is reasonable to assume in addition $\frac{1}{2} \in \mathsf{R}$ since one may pass to an appropriate quotient anyway. Now if $\frac{1}{2} \in \mathsf{R}$ then we can decompose any cochain R-linearly into a Hermitian and anti-Hermitian part, i.e.,

$$\varphi = \frac{1}{2}(\varphi + \varphi^*) + \mathrm{i}\frac{1}{2\mathrm{i}}(\varphi - \varphi^*), \tag{3.5}$$

and an easy check using (3.4) shows that the same holds on the level of cohomology whence we have the following R-linear isomorphism

$$\begin{aligned} H^\bullet(\mathscr{A}) &\cong H^\bullet_H(\mathscr{A}) \oplus \mathrm{i}H^\bullet_H(\mathscr{A}) \\ [\varphi] &\mapsto \left[\tfrac{1}{2}(\varphi + \varphi^*)\right] + \mathrm{i}\left[\tfrac{1}{2\mathrm{i}}(\varphi - \varphi^*)\right]. \end{aligned} \tag{3.6}$$

Now let us consider a deformation $\mu = \sum_{r=0}^{\infty} \lambda^r \mu_r$ of μ_0. Then the associativity condition for μ is well known to be equivalent to $[\mu, \mu] = 0$ which yields $[\mu_0, \mu_0] = 0$ in zeroth order and in higher orders of λ one has

$$\delta\mu_r = -\frac{1}{2}\sum_{s=1}^{r-1}[\mu_s, \mu_{r-s}]. \tag{3.7}$$

To obtain in addition a Hermitian deformation we need

$$\mu_r^* = \mu_r \tag{3.8}$$

for all r since λ is 'real' in $\mathsf{C}[[\lambda]]$. Thus we end up with the following observation that if one can solve both associativity and Hermiticity up to order $r-1$, then the obstructions to solve (3.7) and (3.8) are given by $H^3_H(\mathscr{A})$. If on the other hand one is able to solve the cohomological equation (3.7) for μ_r then there is even a Hermitian solution since an easy computation shows that the right hand side in (3.7) is anti-Hermitian and thus by (3.6) one may simply take the Hermitian part $\frac{1}{2}(\mu_r + \mu_r^*)$ instead of μ_r. Nevertheless note that this is in general not cohomologous to μ_r and thus there may be associative deformations of μ_0 which are *not* equivalent to a Hermitian deformation. This happens e.g. for star-products on symplectic manifolds with non-trivial second deRham cohomology, see [34]. We summarize these results in the following proposition:

Proposition 3.1 *Let $\mathscr{A}$ be a $*$-algebra and let $\mu_0 + \lambda\mu_1 + \ldots + \lambda^{r-1}\mu_{r-1}$ be a Hermitian deformation up to order $r-1$. Then the obstruction space for a Hermitian deformation up to order r is given by $H^3_H(\mathscr{A})$. If there exists an associative deformation up to order r then there also exists an Hermitian deformation up to order r.*

4. Positive deformations

Let us now discuss the main object of this work, the definition of positive deformations of $*$-algebras. In order to motivate this let us first recall the following example of [14, Sect. 2, p. 555]: Consider the phase space $\mathbb{R}^{2n}$ with the canonical coordinates $q^1, \ldots, q^n, p_1, \ldots, p_n$ and the symplectic form $\omega = \sum_k dq^k \wedge dp_k$. Then the Weyl-Moyal star-product, see [3], is given by

$$f *_{\mathrm{Weyl}} g = \mu_0 \circ \mathrm{e}^{\frac{\mathrm{i}\lambda}{2} \sum_k (\partial_{q^k} \otimes \partial_{p_k} - \partial_{p_k} \otimes \partial_{q^k})} f \otimes g, \tag{4.1}$$

where $f, g \in C^\infty(\mathbb{R}^{2n})[[\lambda]]$, and μ_0 is the pointwise product. Clearly $\overline{(f *_{\mathrm{Weyl}} g)} = \overline{g} *_{\mathrm{Weyl}} \overline{f}$ and thus $*_{\mathrm{Weyl}}$ is a Hermitian deformation of the pointwise product with the complex conjugation as $*$-involution. Consider the Hamiltonian $H = \sum_k (q^k q^k + p_k p_k)$ of the harmonic oscillator and let $\delta : C^\infty(\mathbb{R}^{2n})[[\lambda]] \to \mathbb{C}[[\lambda]]$ be the δ-functional at 0 which is classically a positive linear functional. Then

$$\delta(H *_{\mathrm{Weyl}} H) = -\frac{\lambda^2}{2} < 0 \tag{4.2}$$

shows that δ is no longer a positive linear functional for the Weyl-Moyal star-product. On the other hand it is clear that the classical limit of a positive $\mathsf{C}[[\lambda]]$-linear functional of a deformed $*$-algebra $\mathscr{A}[[\lambda]]$ has to be positive with respect to the undeformed $*$-algebra structure. This motivates the following definition:

Definition 4.1 *Let $\mathscr{A}$ be a $*$-algebra over C and let $(\mathscr{A}[[\lambda]], \mu, I)$ be a $*$-algebra deformation of $\mathscr{A}$.*

i) The deformation is called positive if for any classically positive C-linear functional ω_0 there exist C-linear maps $\omega_r : \mathscr{A} \to \mathsf{C}$, $r \geq 1$, such that the $\mathsf{C}[[\lambda]]$-linear functional $\omega = \sum_{r=0}^\infty \lambda^r \omega_r : \mathscr{A}[[\lambda]] \to \mathsf{C}[[\lambda]]$ is positive with respect to the deformed $*$-algebra structure.

ii) The deformation is called strongly positive if any classically positive linear functional is positive for the deformed $*$-algebra, too.

Clearly positivity of a $*$-algebra deformation is a property of the whole corresponding equivalence class of $*$-algebra deformations as one can pull-back positive functionals with the equivalence transformation which yields again a positive functional without changing the classical limit. Nevertheless, this is no longer true for strongly positive deformations in general and it remains to find an appropriate notion of equivalence here.

As a first application we observe that having sufficiently many positive linear functionals is a property which is preserved under positive deformations:

Proposition 4.2 *Let $\mathscr{A}$ be a $*$-algebra over C having sufficiently many positive linear functionals. Then every positive deformation of $\mathscr{A}$ has sufficiently many positive linear functionals.*

Proof. Let $(\mathscr{A}[[\lambda]], \mu, I)$ be a positive deformation of $(\mathscr{A}, \mu_0, I_0)$ and let $A = \sum_{r=r_0}^\infty \lambda^r A_r$ be a non-zero Hermitian element with respect to I with first non-vanishing order r_0. Then $A_{r_0} = I_0(A_{r_0})$ is Hermitian with respect to the classical $*$-involution and thus we find a positive C-linear functional ω_0 of $\mathscr{A}$ with $\omega_0(A_{r_0}) \neq 0$. Moreover, we find $\omega_r : \mathscr{A} \to \mathsf{C}$ such that $\omega = \sum_{r=0}^{\infty} \lambda^r \omega_r$ is a positive $\mathsf{C}[[\lambda]]$-linear functional of $\mathscr{A}[[\lambda]]$ with respect to the deformed $*$-algebra structure. But then clearly $\omega(A) \neq 0$, proving the proposition. ■

As $*$-algebras with sufficiently many positive linear functionals behave in many aspects almost like C^*-algebras we are mainly interested in such positive deformations.

The main difficulty in discussing the existence of (strongly) positive deformations is that one has to deal with inequalities instead of equalities which seems to exclude a suitable cohomological approach. It may even be difficult to decide whether a given $*$-algebra deformation, as e.g. the Weyl-Moyal star-product, is positive or not. Nevertheless in the case of a Hermitian deformation we have the following simple but rather useful criterion:

Lemma 4.3 *Let $(\mathscr{A}[[\lambda]], \mu, I_0)$ be a Hermitian deformation of a $*$-algebra $\mathscr{A}$ over C. If for all $A \in \mathscr{A}$ and $r \geq 1$ one has $\mu_r(A^* \otimes A) \in \mathscr{A}^+$ then the deformation μ is strongly positive.*

Proof. Thanks to [9, Lem. A.5] it is sufficient to check the positivity of a $\mathsf{C}[[\lambda]]$-linear functional $\omega : \mathscr{A}[[\lambda]] \to \mathsf{C}[[\lambda]]$ on elements in $\mathscr{A}$ alone. But if $\omega_0 : \mathscr{A} \to \mathsf{C}$ is a positive linear functional then clearly $\omega_0(\mu(A^* \otimes A)) = \sum_{r=0}^{\infty} \lambda^r \omega_0(\mu_r(A^* \otimes A)) \geq 0$. ■

Based on this lemma we have the following class of strongly positive deformations of $*$-algebras constructed by using commuting derivations:

Lemma 4.4 *Let $\mathscr{A}$ be a $*$-algebra over $\mathsf{C} = \mathsf{R}(\mathrm{i})$, where we assume $\mathbb{Q} \subseteq \mathsf{R}$, and let $D_1, \ldots, D_n$ be derivations of $\mathscr{A}$ such that $[D_i, D_j] = 0 = [D_i, D_j^*]$ for all $i, j = 1, \ldots, n$. Then*

$$\mu = \mu_0 \circ \mathrm{e}^{\lambda \sum_k D_k \otimes D_k^*} \tag{4.3}$$

defines a strongly positive Hermitian deformation of $\mathscr{A}$.

Proof. First we note that the exp-series is well defined thanks $\mathbb{Q} \subseteq \mathsf{R}$. As already observed , see (3.4), the C-linear maps D_i^* are again derivations and thus all occuring derivations commute which implies that (4.3) is associative. Clearly μ is Hermitian and one has $\mu_r(A^* \otimes A) \in \mathscr{A}^{++} \subseteq \mathscr{A}^+$ for all $r \geq 1$ whence we can apply the above lemma. ■

5. Positive deformations in deformation quantization

Let us now discuss the situation in deformation quantization. The main example for a deformation of the type as in Lemma 4.4 is the Wick product $*_{\mathrm{Wick}}$ for $\mathbb{C}^n$: denote by $z^1, \ldots, z^n, \overline{z}^1, \ldots, \overline{z}^n$ the global (anti-) holomorphic coordinates then $*_{\mathrm{Wick}}$ is defined for $f, g \in C^\infty(\mathbb{C}^n)[[\lambda]]$ by

$$f *_{\mathrm{Wick}} g = \mu_0 \circ \mathrm{e}^{2\lambda \sum_k \partial_{z^k} \otimes \partial_{\overline{z}^k}} f \otimes g, \tag{5.1}$$

which is clearly of the above form and thus strongly positive and Hermitian. See [27, 28, 29, 12] for a more general treatment of such star-products with separation of variables on Kähler manifolds. From the above remark on equivalent Hermitian deformations and the well-known fact that on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ all star-products are equivalent we conclude that the Weyl-Moyal star-product $*_{\mathrm{Weyl}}$ is an example of a positive but *not* strongly positive deformation. More explicitly, we have the following: Using the explicit (and real) equivalence transformation $T = \mathrm{e}^{\lambda\Delta}$, where $\Delta = \sum_k \partial_{z^k} \partial_{\overline{z}^k}$ is the Laplacian and $f *_{\mathrm{Weyl}} g = T^{-1}(Tf *_{\mathrm{Wick}} Tg)$, we conclude that for any classically positive linear functional ω_0 the $\mathbb{C}[[\lambda]]$-linear functional $\omega = \omega_0 \circ T = \omega_0 \circ \mathrm{e}^{\lambda\Delta}$ is positive with respect to $*_{\mathrm{Weyl}}$. In particular, one obtains a deformation $\delta \circ \mathrm{e}^{\lambda\Delta}$ of the delta-functional which is positive. But note also that for some particular ω_0 there may be no need to deform it in order to obtain positivity, see e.g. [8, 9].

A local version of the above construction can even be used to show that all Hermitian star-products on a symplectic manifold are positive deformations:

Proposition 5.1 *Let $(M, *)$ be a symplectic manifold with a Hermitian star product. Then $*$ is a positive deformation.*

Proof. Let $\omega_0 : C_0^\infty(M) \to \mathbb{C}$ be a classically positive linear functional. Choose a locally finite open cover $\{O_\alpha\}_{\alpha \in I}$ of M by contractable charts and let $\{\chi_\alpha\}_{\alpha \in I}$ be a subordinate 'quadratic' partition of unity, i.e., $\sum_\alpha \overline{\chi}_\alpha \chi_\alpha = 1$. Endow each O_α with a local Wick star-product $*_\alpha$ and let T_α be a local and real equivalence transformation between $*|_{O_\alpha}$ and $*_\alpha$, i.e., $T_\alpha(f * g) = T_\alpha f *_\alpha T_\alpha g$ on O_α. Such an equivalence transformation exists since O_α is contractable, see e.g. [4]. Then it is easy to check that the $\mathbb{C}[[\lambda]]$-linear functional $\omega : C_0^\infty(M)[[\lambda]] \to \mathbb{C}[[\lambda]]$ defined by

$$\omega(f) := \sum_\alpha \omega_0 (T_\alpha (\overline{\chi}_\alpha * f * \chi_\alpha)) \tag{5.2}$$

is well defined, positive with respect to $*$, and a deformation of ω_0. ■

Since $C_0^\infty(M)$ (or $C^\infty(M)$, respectively) has sufficiently many positive linear functionals we obtain from Propositions 4.2 and 5.1 the following useful corollary:

Corollary 5.2 *Let $(M, *)$ be a symplectic manifold with a Hermitian star product. Then the $\mathbb{C}[[\lambda]]$-algebra $(C_0^\infty(M)[[\lambda]], *)$ has sufficiently many positive $\mathbb{C}[[\lambda]]$-linear functionals.*

The case of a Hermitian star-product on an arbitrary Poisson manifold seems to be more involved since the proof of Proposition 5.1 uses at several points that the Poisson structure is symplectic. Thus it remains an open question whether or which star-products on arbitrary Poisson manifolds are positive deformations. Nevertheless we can show directly the weaker property that a Hermitian star-product on a Poisson manifold has sufficiently many positive linear functionals.

Proposition 5.3 *Let $(M, *)$ be a Poisson manifold with a Hermitian star-product. Then the $\mathbb{C}[[\lambda]]$-algebra $(C_0^\infty(M)[[\lambda]], *)$ has sufficiently many positive $\mathbb{C}[[\lambda]]$-linear functionals.*

Proof. Let $0 \neq f = \sum_{r=0}^\infty \lambda^r f_r \in C_0^\infty(M)[[\lambda]]$ be Hermitian then we may assume that $f_0 \neq 0$. Thus we find a point $x \in M$ and an open neighborhood $U \subseteq M$ of x such that $f|_U$ is either strictly positive or strictly negative. Choose now a smooth density $\mu \in \Gamma_0^\infty(|\bigwedge^n| T^*M)$ with $\mu(x) > 0$, $\mu \geq 0$, and $\operatorname{supp} \mu \subset U$. Then it follows by the same argument as in [14, Lem. 2] that the functional $\omega : g \mapsto \int_M g\mu$ is positive with respect to any star product on M and clearly we have $\omega(f) \neq 0$. ■

Let us conclude with a few remarks: It is clear that if a deformation ω of a classically positive linear functional ω_0 into a positive $\mathsf{C}[[\lambda]]$-linear functional of a deformed algebra $\mathscr{A}[[\lambda]]$ exists, the higher orders ω_r of the deformation are not necessarily unique. If for example the functional ω_0 is faithful, i.e., if $\omega_0(A^*A) > 0$ for all $A \neq 0$, then one can add *any* real linear functionals ω_r in higher orders and not lose positivity of ω. This raises the question of whether in a particular case there are in some sense 'natural' or 'minimal' corrections ω_r which deform ω_0 into a positive linear functional. In the case of star-products one can impose further conditions on ω_r by requiring e.g. continuity with respect to the canonical locally convex topologies of smooth functions. Moreover, one can demand that the corrections should not increase the support of the classical functional ω_0, see [40] for an extensive discussion on the involved locality structures. But even then a tremendous variety remains, since one can pull-back deformed positive linear functionals by $*$-automorphisms of the deformed algebra with the identity as classical limit. Then one ends up again with a deformation of the same classically positive linear functional. Hence a classification of positive deformations of a given classically positive linear functional modulo the above action of $*$-automorphisms would be highly desirable.

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Symmetric symplectic spaces with Ricci-type curvature

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Abstract. We determine the isomorphism classes of symmetric symplectic manifolds of dimension at least 4 which are connected, simply-connected and have a curvature tensor which has only one non-vanishing irreducible component — the Ricci tensor.

Keywords: symplectic connection, symmetric space

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To Moshé

Introduction

On any symplectic manifold (M,ω) the space of symplectic connections (linear connections ∇ with vanishing torsion and such that $\nabla\omega = 0$) is infinite dimensional. In order to select a smaller family of symplectic connections, a variational principle was introduced in [2]. This principle has Euler-Lagrange equations

$$(\nabla_X r)(Y,Z) + (\nabla_Y r)(Z,X) + (\nabla_Z r)(X,Y) = 0 \qquad (0.1)$$

for all vector fields X,Y,Z; r denotes the Ricci tensor of ∇

$$r(X,Y) = \mathrm{Tr}(Z \mapsto R(X,Z)Y).$$

In [2] the case where $\dim M = 2$ was examined in complete detail so we shall assume throughout that the dimension of M is at least 4.

It was observed in [3] that the field equations (0.1) are identically satisfied if one assumes that the irreducible component of the curvature, denoted there by W

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(see also [5]), vanishes

$$W = 0. \tag{0.2}$$

The tensor W is the symplectic analogue of the Weyl or conformal curvature of a Riemannian connection. The vanishing of W (Equation (0.2)) is equivalent to the requirement that the curvature tensor R of ∇ is expressed in terms of its Ricci tensor by

$$\begin{aligned} R(X,Y)Z = \frac{1}{2(n+1)} \big[& 2\omega(X,Y)AZ + \omega(X,Z)AY - \omega(Y,AZ)X \\ & - \omega(Y,Z)AX + \omega(X,AZ)Y \big] \end{aligned} \tag{0.3}$$

where $\dim M = 2n$, $n \geq 2$, where X, Y, Z are vector fields and where A is the Ricci tensor viewed as an endomorphism of the tangent bundle using ω:

$$r(X,Y) = \omega(X,AY). \tag{0.4}$$

The Ricci tensor is symmetric so A is an infinitesimal symplectic endomorphism of each tangent space.

Equation (0.2) (or (0.3)) implies the existence of a 1-form u on (M,ω) such that

$$(\nabla_X r)(Y,Z) = \omega(X,Y)u(Z) + \omega(X,Z)u(Y). \tag{0.5}$$

If $u = 0$, then $\nabla r = 0$ and since R is expressed in terms of r (0.3), ∇ is locally symmetric.

The condition $W = 0$ also appears as the integrability condition for the almost complex structure naturally defined from a symplectic connection on (M,ω) on the manifold $\mathcal{J}(M)$ of almost complex stuctures on M which are compatible with ω.

In this note we prove, amongst other things, the following two results.

Theorem 1 *Let* $(M,\omega) = (M_1,\omega_1) \times (M_2,\omega_2)$ *be symplectic manifolds of dimension greater than zero and* $\nabla = \nabla_1 + \nabla_2$ *be a symplectic connection. If* $W^\nabla = 0$ *then* $\nabla, \nabla_1, \nabla_2$ *are flat.*

Theorem 2 *Let* (M,ω,s) *be a connected, simply-connected, symmetric symplectic space of dimension* $2n$ (≥ 4)*; let* ∇ *be its canonical invariant symplectic connection and let* r *be its Ricci tensor; let* A *be the corresponding endomorphism*

$$\omega(X,AY) = r(X,Y).$$

Assume $W^\nabla = 0$. *Then*

$$A^2 = \lambda \, \mathrm{Id}$$

for some real number λ.

If $\lambda \neq 0$, *the transvection group* G *of* (M,ω,s) *is semisimple and, up to coverings,* $M = G/K$ *with either* $G = SL(n+1,\mathbb{R})$ *and* $K = GL(n,\mathbb{R})$ *or* $G = SU(p+1,q)$ *and* $K = U(p,q)$ *where* $\dim M = 2n$, $p+q = n$.

If $\lambda = 0$ *and* $\operatorname{Rank}(A) > 1$, *the transvection group* G *of* (M, ω, s) *is neither solvable nor semisimple. The radical of* G *is 2-step unipotent if* $\operatorname{Rank}(A) < n$ *and abelian in* $\operatorname{Rank}(A) = n$. *If* $\lambda = 0$ *and* $\operatorname{Rank}(A) = 1$, *the transvection group* G *of* (M, ω, s) *is solvable.*

1. Proof of Theorem 1

Let $(M, \omega) = (M_1, \omega_1) \times (M_2, \omega_2)$ be symplectic manifolds and $\nabla = \nabla_1 + \nabla_2$ be a symplectic connection. Then $R(X,Y)Z = R_1(X_1,Y_1)Z_1 + R_2(X_2,Y_2)Z_2$ where $X = X_1 + X_2$, $Y = Y_1 + Y_2$, $Z = Z_1 + Z_2$ and suffices indicate components tangent to M_1 and M_2, respectively. Then also $r(X,Y) = r_1(X_1,Y_1) + r_2(X_2,Y_2)$. On the other hand, the relation between W, W_1 and W_2 involves cross terms $C(X,Y)Z$:

$$W(X,Y)Z = W_1(X_1,Y_1)Z_1 + W_2(X_2,Y_2)Z_2 + C(X,Y)Z.$$

These can be read off Equation (0.3). Then $W = 0$ implies $W_1 = 0$, $W_2 = 0$ and $C = 0$. We have

$$C(X_1,Y_1)Z_2 = \frac{1}{2(n+1)}\left[-2\omega(X_1,Y_1)A_2Z_2\right]$$

so $A_2 = 0$ and interchanging 1 and 2 we see also $A_1 = 0$. Thus $r_1 = 0$ and $r_2 = 0$, and hence $R_1 = 0$ and $R_2 = 0$.

2. Some facts about symmetric symplectic spaces

Affine symmetric spaces are studied in Loos [4], symplectic symmetric spaces are studied in Bieliavsky [1].

Definition 3 A **symmetric symplectic manifold** is a triple (M, ω, s) where M is a smooth connected manifold, where ω is a smooth symplectic form on M and where s is a smooth map $M \times M \to M$, $(x,y) \mapsto s_x(y)$, such that:

(i) For each x in M, s_x is an involutive symplectic diffeomorphism of (M, ω) (called the symmetry at x) and x is an isolated fixed point of s_x;

(ii) $s_x s_y s_x = s_{s_x(y)}$ for all x, y in M.

The **transvection group** G of (M, ω, s) is the group generated by products of an even number of symmetries.

We recall below some general facts about symmetric spaces ([4], [1]).

(1) (M, ω, s) has a unique connection ∇ such that $\nabla\omega = 0$ and such that each symmetry s_x is an affine transformation of (M, ∇). Observe that $s_{x*x} = -\operatorname{Id}_{T_xM}$

because $(s_{x*x})^2 = \mathrm{Id}_{T_xM}$ and x is an isolated fixed point of s_x. Since $\omega_x(\nabla_X Y, Z) = \frac{1}{2}(\omega_x(\nabla_X Y, Z) + (s_x{}^*\omega)_x(\nabla_X Y, Z))$, the connection is given by

$$\omega_x(\nabla_X Y, Z) = \frac{1}{2} X_x(\omega(Y + s_x \cdot Y, Z)) \tag{2.1}$$

for $x \in M$, where X, Y, Z are vector fields on M and $(s_x \cdot Y)_y = s_{x*} Y_{s_x(y)}$. This connection ∇ has no torsion and is thus a symplectic connection. The symmetry s_x coincides with the geodesic symmetry around x, since an affinity is determined by its 1-jet at a point.

(2) The automorphism group $\mathrm{Aut} = \mathrm{Aut}(M, \omega, s)$ of (M, ω, s) is the set of symplectic automorphisms φ of (M, ω) such $\varphi \circ s_x = s_{\varphi(x)} \circ \varphi$, $\forall x \in M$. It is the intersection of the affine group of (M, ∇) and the symplectic diffeomorphism group of (M, ω). It is thus a Lie group containing the transvection group so acts transitively on M (since any two points in M can be joined by a broken geodesic).

Choose a base point o in M. Denote by $\widetilde{\sigma}$ the conjugation by the symmetry s_o, it is an involutive automorphism of Aut.

Let K' denote the stabilizer of o in Aut and let $A^{\widetilde{\sigma}}$ (respectively $A_o^{\widetilde{\sigma}}$) denote the group of fixed points of $\widetilde{\sigma}$ in Aut (respectively its connected component). Then $A^{\widetilde{\sigma}} \supseteq K' \supseteq A_o^{\widetilde{\sigma}}$.

Hence, if $\mathfrak{a}$ (respectively $\mathfrak{k}'$) is the Lie algebra of Aut (respectively K') and if $\sigma = \widetilde{\sigma}_{*Id}$, then $\mathfrak{k}'$ is the subalgebra of $\mathfrak{a}$ of fixed points of σ.

(3) Let $\mathfrak{p} = \{X \in \mathfrak{a} \mid \sigma(X) = -X\}$. Then $\mathfrak{a} = \mathfrak{k}' \oplus \mathfrak{p}$.

Denote by π' the projection $\mathrm{Aut} \to M$ given by $\pi'(g) = g \cdot o$. Then $\pi'_{*e}|_{\mathfrak{p}} : \mathfrak{p} \to T_oM$ is a linear isomorphism which identifies the tangent space T_oM with $\mathfrak{p}$.

Denote by $\mathrm{Exp}\colon T_oM \to M$ the exponential map given by the connection ∇ at the point o and by exp the exponential map from the Lie algebra $\mathfrak{a}$ to the Lie group Aut.

Observe that $s_{\mathrm{Exp}\frac{t}{2}X} s_o$, $X \in T_oM$, is an affinity in G which realises the parallel transport along $\mathrm{Exp}\, tX$, since $s_{\mathrm{Exp}\, uX*}$ for any $u \in \mathbb{R}$ maps a vector field which is parallel along the geodesic $\mathrm{Exp}\, tX$ to another such parallel vector field. Hence $s_{\mathrm{Exp}\frac{t}{2}\pi'_{*e}X} s_o = \exp tX$, $\forall X \in \mathfrak{p}$.

It follows that the transvection group G, which is stable by $\widetilde{\sigma}$, is the connected Lie subgroup of $\mathrm{Aut}(M, \omega, s)$ whose Lie algebra is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \qquad \text{where} \qquad \mathfrak{k} = [\mathfrak{p}, \mathfrak{p}]. \tag{2.2}$$

Indeed, if G_1 denotes that subgroup, clearly by the above $G_1 \subset G$ and the parallel transport along a geodesic $\mathrm{Exp}\, tX$ is in G_1; but then any $x \in M$ can be written as $x = g \cdot o$ for $g \in G_1$ hence $s_x s_o = g s_o g^{-1} s_o = g\widetilde{\sigma}(g^{-1}) \in G_1$ and $G \subset G_1$.

Let K denote the stabilizer of o in G. Its Lie algebra is $\mathfrak{k}$ and $\mathfrak{k} = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$. Since the Lie group G acts effectively on M, the representation of K on T_oM, $k \mapsto k_{*o}$, is faithful so $\mathfrak{k}$ acts faithfully on $\mathfrak{p}$.

(4) Denote by π the projection $\pi\colon G \to M$ where $\pi(g) = g \cdot o$. Denote by X^* the vector field on M which is the image under π_* of the right invariant vector field on G, i.e., ${X^*}_{g\cdot o} = \frac{d}{dt} \exp tX \cdot g \cdot o|_{t=0}$. Observe that $[X^*, Y^*] = -[X,Y]^*$. Since ω is invariant under G, formula (2.1) yields $\omega_x(\nabla_{Y^*}X^*, Z^*) = \frac{1}{2}\omega_x([Y^*, X^* + s_x \cdot X^*], Z^*)$ so $(\nabla_{X^*}Y^*)_x = [X^*, Y^*] + \frac{1}{2}[Y^*, X^* + s_x \cdot X^*]$. But $s_{g\cdot o} \cdot X^* = g \cdot s_o \cdot g^{-1} \cdot X^* = (\mathrm{Ad}\, g\sigma(\mathrm{Ad}\, g^{-1}X))^*$ so the connection has the form

$$(\nabla_{X^*}Y^*)_{g\cdot o} = ([Y, \mathrm{Ad}\, g(\mathrm{Ad}\, g^{-1}X)_{\mathfrak{p}}])^*_{g\cdot o} \tag{2.3}$$

where $Z_{\mathfrak{p}}$ denotes the component in $\mathfrak{p}$ of $Z \in \mathfrak{g}$ relatively to the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ and where $[\ ,\ ]$ is the bracket in $\mathfrak{g}$.

Since any G-invariant tensor on M is parallel, the curvature tensor of (M, ∇) is parallel ($\nabla R = 0$) and if X, Y, Z belong to $\mathfrak{p}$, one has,

$$R_o(X^*_o, Y^*_o)Z^*_o = -([[X,Y],Z])^*_o. \tag{2.4}$$

Definition 4 A **symmetric symplectic triple** is a triple $(\mathfrak{g}, \sigma, \Omega)$ where $\mathfrak{g}$ is a finite dimensional real Lie algebra, σ is an involutive automorphism of $\mathfrak{g}$ such that if we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\sigma = \mathrm{Id}_{\mathfrak{k}} \oplus -\mathrm{Id}_{\mathfrak{p}}$, then

- $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$;
- The action of $\mathfrak{k}$ on $\mathfrak{p}$ is faithful;

and where Ω is a non degenerate skewsymmetric 2-form on $\mathfrak{p}$, invariant by $\mathfrak{k}$ under the adjoint action.

We have seen above that to any connected symmetric symplectic manifold (M, ω, s), when one chooses a base point $o \in M$, one associates a symmetric symplectic triple $(\mathfrak{g}, \sigma, \Omega)$ with $\mathfrak{g}$ the Lie algebra of its transvection group, with σ the differential at the identity of the conjugation by the symmetry s_o and with $\Omega = \omega_o$ with the identification between T_oM and $\mathfrak{p}$.

Reciprocally, given a symmetric symplectic triple $(\mathfrak{g}, \sigma, \Omega)$, one builds a simply-connected symmetric symplectic space (M, ω, s) with $M = G/K$ where G is the simply-connected Lie group with Lie algebra $\mathfrak{g}$ and K is its connected subgroup with Lie algebra $\mathfrak{k}$, with ω the G-invariant 2-form on M whose value at eK is given by Ω (identifying $T_{eK}M$ and $\mathfrak{p}$ via the differential of the canonical projection $\pi\colon G \to G/K$) and with symmetries defined by $s_{\pi(g)}\pi(g') = \pi(g\widetilde{\sigma}(g^{-1}g'))$ where $\widetilde{\sigma}$ is the automorphism of G whose differential at e is σ.

3. Proof of Theorem 2

Consider a symmetric symplectic space (M,ω,s) and assume that its canonical invariant symplectic connection ∇ has a curvature with $W=0$.

Since $\nabla R=0$, the Ricci tensor r and its associated endomorphism A (where $r(X,Y)=\omega(X,AY)$) are covariantly constant and hence A commutes with the curvature endomorphisms

$$AR(X,Y)=R(X,Y)A.$$

This implies, when we substitute R by its expression in terms of A into Equation (0.3)

$$-\omega(X,Z)A^2Y+\omega(Y,Z)A^2X=\omega(Y,A^2Z)X-\omega(X,A^2Z)Y.$$

If $Y\neq 0$ is arbitrary, $Z=Y$, and we pick X so that $\omega(X,Y)=1$, then $\omega(Y,A^2Y)=\omega(AY,AY)=0$, so $A^2Y=\lambda_Y Y$ for some function λ_Y. Substituting back into the equation shows that $\lambda_Y=\lambda$ is independent of Y, and since A is covariant constant, λ must be constant.

Remark that if $\lambda\neq 0$ then r is a non-degenerate parallel symmetric bilinear form so ∇ is its Levi-Civita connection and (M,r,s) is a pseudo-Riemannian symmetric space.

Let G be the transvection group of our symmetric symplectic space. Choose a base point $o\in M$ and let $(\mathfrak{g},\sigma,\Omega)$ be the symmetric triple associated to (M,ω,s). Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the decomposition of the Lie algebra of G into the $+1$ and -1 eigenspaces of σ. Then $\Omega(X,Y)=\omega_o(X^*_o,Y^*_o)$ and with a slight abuse of notations we denote by R the map $R\colon\mathfrak{p}\times\mathfrak{p}\to\operatorname{End}(\mathfrak{p})$ so that $(R(X,Y)Z)^*_o=R_o(X^*_o,Y^*_o)Z^*_o$ and by A the map $A\colon\mathfrak{p}\to\mathfrak{p}$ so that $(A(X))^*_o=A_o(X^*_o)$. Since $\mathfrak{k}$ acts faithfully on $\mathfrak{p}$, we view $\mathfrak{k}$ as a subset of $\operatorname{End}(\mathfrak{p})$; by formula (2.4),

$$\mathfrak{k}=\{R(X,Y)\in\operatorname{End}(\mathfrak{p})\mid X,Y\in\mathfrak{p}\}\tag{3.1}$$

and the brackets on $\mathfrak{g}\subset\mathfrak{p}\oplus\operatorname{End}(\mathfrak{p})$ are

$$[(C,X),(D,Y)]=([C,D]-R(X,Y),CY-DX)\tag{3.2}$$

where $C,D\in\mathfrak{k}\subset\operatorname{End}(\mathfrak{p})$, and $X,Y\in\mathfrak{p}$.

Define the 1-form on $\mathfrak{p}$ corresponding to a vector $X\in\mathfrak{p}$ by $\underline{X}=i(X)\Omega$. Formula (0.3) giving the curvature when $W=0$ is equivalent to

$$R(X,Y)=k\,(2\Omega(X,Y)A+AY\otimes\underline{X}-AX\otimes\underline{Y}+X\otimes\underline{AY}-Y\otimes\underline{AX})$$

where $k=1/(m+2)$ if $m=\dim M=2n$. Note that, for a symplectic symmetric space built from a Lie algebra $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ whose bracket of $\mathfrak{p}$ into $\mathfrak{k}$ is given by this formula, then the canonical connection will have curvature given by this formula and hence W will vanish.

Define $B = Y \otimes \underline{X} - X \otimes \underline{Y}$. Clearly B satisfies $\Omega(U,BV) = \Omega(BU,V)$ and any antisymplectic endomorphism of $\mathfrak{p}$ can be written as a sum of such operators. Then

$$R(X,Y) = k(\mathrm{Tr}(B)A + AB + BA)$$

and, if we put $B' = k(B + \frac{1}{2}\mathrm{Tr}(B)I)$, the RHS becomes $C = AB' + B'A$.

Lemma 5 *For any λ,*

$$\mathfrak{k} = \{C = AB + BA \mid B \in \mathrm{End}(\mathfrak{p}) \text{ and } \Omega(X,BY) = \Omega(BX,Y)\}.$$

If $\lambda \neq 0$ then $\mathfrak{k}$ is the set of endomorphisms $C \in \mathrm{End}(\mathfrak{p})$ which are infinitesimally symplectic and commute with A.

Proof. The first part follows from the considerations above and the fact that the map $B \mapsto B + \frac{1}{2}\mathrm{Tr}(B)$ is a bijection on the space of antisymplectic endomorphisms of $\mathfrak{p}$. C commutes with A since $AC = \lambda B' + AB'A = CA$. Also $\Omega(X,CY) = -\Omega(AX,B'Y) + \Omega(B'X,AY) = -\Omega(B'AX,Y) - \Omega(AB'X,Y) = -\Omega(CX,Y)$.

Conversely, if $\lambda \neq 0$, given C commuting with A and such that $\Omega(X,CY) = -\Omega(CX,Y)$, let $B = \frac{1}{2}\lambda^{-1}AC$; then

$$BA + AB = \frac{1}{2}\lambda^{-1}2\lambda C = C.$$

■

3.1. Case $\lambda > 0$

Write $\lambda = a^2$, $a > 0$. Then $\mathfrak{p} = V^+ \oplus V^-$ where $V^\pm = \{X \in \mathfrak{p} \mid AX = \pm aX\}$. Let $P^\pm$ be the projection onto $V^\pm$. Then $A = a(P^+ - P^-)$. Clearly

$$\begin{aligned}
\omega(V^+,V^+) &= \omega(V^-,V^-) = 0,\\
R(V^+,V^+) &= R(V^-,V^-) = 0,\\
R(X,Y) &= 2ka(\Omega(X,Y)(P^+ - P^-) - Y \otimes \underline{X} - X \otimes \underline{Y}),
\end{aligned}$$

for $X \in V^+, Y \in V^-$. It follows that $V^\pm$ are Lagrangian subspaces of $\mathfrak{p}$. Identifying V^- with $(V^+)^*$ via $Y \mapsto \underline{Y}|_{V^+}$ and renaming V^+ as V, we have identified $\mathfrak{p}$ with $V \oplus V^*$ with its standard symplectic structure $\Omega(X + \xi, X' + \xi') = -\langle X, \xi' \rangle + \langle X', \xi \rangle$, and A acts as $+a$ on V, $-a$ on V^*. With this notation the curvature has the form

$$R(X,\xi) = 2ak(-\langle X,\xi \rangle(\mathrm{Id}_V - \mathrm{Id}_{V^*}) + \xi \otimes X - X \otimes \xi).$$

The symplectic centraliser of A can then be identified with $\mathrm{End}(V) = \mathfrak{gl}(V)$, identifying the element in $\mathrm{End}(\mathfrak{p}) = \mathrm{End}(V \oplus V^*)$ given by

$$\begin{pmatrix} C & 0 \\ 0 & -{}^tC \end{pmatrix}$$

with the element $C \in \mathfrak{gl}(V)$.

So $\mathfrak{k} = \mathfrak{gl}(V)$ and as a vector space $\mathfrak{g} = \mathfrak{gl}(V) \oplus V \oplus V^*$ with the brackets

$$\begin{aligned}[(C,X,\xi),(C',X',\xi')] = ([C,C'] &+ 2ka(\langle X,\xi'\rangle - \langle X',\xi\rangle)I \\ &+ 2kaX \otimes \xi' - 2kaX' \otimes \xi, \\ &CX' - C'X, -{}^tC\xi' + {}^tC'\xi).\end{aligned}$$

The map $j\colon \mathfrak{g} \to \mathfrak{sl}(V \oplus \mathbb{R})$ given by

$$j(C,X,\xi) = \begin{pmatrix} C - 2k\operatorname{Tr}(C)I & sX \\ s\,{}^t\xi & -2k\operatorname{Tr}(C) \end{pmatrix}$$

has the brackets above provided $s^2 = 2ka$.

Thus when $\lambda > 0$, $M = G/K$ where $G = SL(n+1,\mathbb{R})$, $K = GL(n,\mathbb{R})$. The involution σ is given by

$$\sigma\begin{pmatrix} C & v \\ \xi & -\operatorname{Tr}(C) \end{pmatrix} = \begin{pmatrix} C & -v \\ -\xi & -\operatorname{Tr}(C) \end{pmatrix}$$

and, writing (X,ξ) for $\begin{pmatrix} 0 & X \\ \xi & 0 \end{pmatrix}$, the symplectic form is given by

$$\Omega\left((X,\xi),(X',\xi')\right) = -\langle X,\xi'\rangle + \langle X',\xi\rangle.$$

The curvature of the canonical connection on this symplectic symmetric space at the base point eK is

$$\begin{aligned}R((X,\xi),(X',\xi'))(X'',\xi'') = (X''(\langle X',\xi\rangle - \langle X,\xi'\rangle) - X\langle X'',\xi'\rangle \\ +X'\langle X'',\xi\rangle, \xi'\langle X,\xi''\rangle - \xi\langle X',\xi''\rangle - \xi''(\langle X',\xi\rangle - \langle X,\xi'\rangle)),\end{aligned}$$

$$r((X,\xi),(X',\xi')) = (n+1)(\langle X,\xi'\rangle + \langle X',\xi\rangle),$$

$$A(x,\xi) = (n+1)(x,-\xi),$$

and formula (0.3) holds so R is of Ricci-type.

3.2. Case $\lambda < 0$

We write $\lambda = -b^2$ where $b < 0$. If we put $J = b^{-1}A$ then J defines a complex structure on the vector space $\mathfrak{p}$. We write V for $\mathfrak{p}$ viewed as an n-dimensional complex vector space. V has a (pseudo-)Hermitean structure given by

$$\langle X,Y\rangle = \Omega(X,JY) + i\Omega(X,Y)$$

which is $\mathbb{C}$-linear in the second variable. The infinitesimally symplectic transformations which commute with A, or equivalently J, are the complex linear transformations of V which are skew-Hermitean with respect to this Hermitean structure. Thus $\mathfrak{k}$ is the (pseudo-) unitary Lie algebra $\mathfrak{u}(V,\langle\, ,\, \rangle)$.

The curvature has the form

$$R(X,Y) = kb\,(2\Omega(X,Y)J + Y \otimes \langle X,.\rangle - X \otimes \langle Y,.\rangle).$$

Then $\mathfrak{g} = \mathfrak{u}(V,\langle\,,\,\rangle) \oplus V$ with bracket

$$\begin{aligned}[(C,X),(C',X')] \;=\; &([C,C'] + kb(X \otimes \langle X',.\rangle - X' \otimes \langle X,.\rangle \\ &-2\Omega(X,X')J), CX' - C'X).\end{aligned}$$

and $\mathfrak{g}$ can be identified with $\mathfrak{su}(V \oplus \mathbb{C}, \langle\langle\,,\,\rangle\rangle)$ via

$$j(C,X) = \begin{pmatrix} C - 2k\,\mathrm{Tr}(C)I & sX \\ -\bar{s}\langle X,.\rangle & -2k\,\mathrm{Tr}(C) \end{pmatrix}$$

with

$$\langle\langle (v,r),(w,t)\rangle\rangle = \langle v,w\rangle + \bar{r}t$$

provided

$$s\bar{s} = -kb.$$

Hence when $\lambda < 0$ then $M = G/K$ with $\mathfrak{g} = \mathfrak{su}(p+1,q)$, $p+q = n$, $\mathfrak{k} = \mathfrak{u}(p,q)$,

$$\sigma\begin{pmatrix} C & v \\ -\langle v,.\rangle & -\mathrm{Tr}(C) \end{pmatrix} = \begin{pmatrix} C & -v \\ \langle v,.\rangle & -\mathrm{Tr}(C) \end{pmatrix}$$

where $\langle v,w\rangle = \sum_{i=1}^{p} \bar{v}^i w^i - \sum_{j=p+1}^{n} \bar{v}^j w^j$ and

$$\Omega(v,w) = \mathrm{Im}\langle v,w\rangle.$$

The curvature of the canonical connection on this symmetric symplectic space at eK is

$$\begin{aligned} R(v,w)z =& v\langle w,z\rangle - w\langle v,z\rangle + z(-\langle v,w\rangle + \langle w,v\rangle), \\ r(v,z) =& -2(n+1)\,\mathrm{Re}\langle v,z\rangle, \\ A(v) =& -2(n+1)iv, \end{aligned}$$

and formula (0.3) holds so R is of Ricci-type.

3.3. Case $\lambda = 0$

In this case A is nilpotent since $A^2 = 0$. Let $Z = \mathrm{Image}\,A$ and $\widetilde{Z} = \mathrm{Ker}\,A$. Then $Z \subset \widetilde{Z}$, and Z and $\widetilde{Z}$ are symplectic orthogonals of each other. If V denotes a complement for Z in $\widetilde{Z}$, then the restriction of Ω to V is non-degenerate. Z is contained in the Ω-orthogonal of V; let Z' be a complement so that $V^{\perp} = Z \oplus Z'$. $V^{\perp}$ is a symplectic subspace and Z is maximal isotropic so we can also suppose that Z' is maximal isotropic. Ω gives a duality of Z with Z'.

In other words, we have written $\mathfrak{p}$ as $Z \oplus Z^* \oplus V$ where $Z \oplus Z^*$ has its standard symplectic structure and V is a symplectic vector space. A is non-zero only on Z^*

and maps it isomorphically onto Z, and as such it is symmetric. In block form, the symplectic structure Ω is given by

$$\begin{pmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & J' \end{pmatrix}$$

and A by

$$\begin{pmatrix} 0 & A' & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where A', by a suitable choice of basis, is diagonal with ± 1 on the diagonal. An easy calculation shows that matrices of the form $AB+BA$ with $\Omega(X,BY) = \Omega(BX,Y)$ have the form

$$\begin{pmatrix} K & L & -{}^tMJ' \\ 0 & -{}^tK & 0 \\ 0 & M & 0 \end{pmatrix}$$

where ${}^tKA' + A'K = 0$, ${}^tL = L$. The matrices with $K = 0$ form an ideal which is 2-step nilpotent (abelian when $\operatorname{Rank} A = n = \frac{1}{2}\dim M$) and the matrices with $L = M = 0$ a subalgebra isomorphic to $\mathfrak{so}(p,q)$, where $p+q = r = \operatorname{Rank} A$, p the number of $+$'s and q the number of $-$'s in A' (hence (p,q) is the signature of the non degenerate symmetric bilinear form naturally induced on $\mathfrak{p}/\operatorname{Ker} A$ by the Ricci tensor $\Omega(X,AY)$).

The bracket of $\mathfrak{p}$ into $\mathfrak{k}$ is given, using formulas (3.2) and (0.3) by

$$\left[\begin{pmatrix} u \\ v \\ w \end{pmatrix}, \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}\right] = -k \begin{pmatrix} \tilde{K} = A'(v'^tv - v^tv') & \tilde{L} & -t\tilde{M}J') \\ 0 & -{}^t\tilde{K} & 0 \\ 0 & \tilde{M} & 0 \end{pmatrix}$$

where $\tilde{L} = A'B + {}^tBA' + 2(\operatorname{Tr} B + {}^twJ'w')A'$ with $B = v^tu' - v'^tu$ and $\tilde{M} = -{}^t(A'(v'^tw - v^tw'))$.

Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \{(K,L,M,u,v,w) \mid K \in \mathfrak{so}(p,q), L \in \operatorname{Mat}(r\times r,\mathbb{R}), {}^tL = L, M \in \operatorname{Mat}(2n-2r\times r,\mathbb{R}), u \in Z = \mathbb{R}^r, v \in Z^*, w \in W = \mathbb{R}^{2n-2r}\}$. The brackets are given, with obvious notations, by

$$[(K,L,M),(K',L',M')] = ([K,K'],L'',-M^tK'+M'^tK)$$

where $L'' = KL' - L^tK' - K'L + L'^tK - {}^tMJ'M' + {}^tM'J'M$,

$$\begin{aligned} [(K,L,M),(u,v,w)] &= (Ku + Lv - {}^tMJ'w, -{}^tKv, Mv), \\ [(u,v,w),(u',v',w')] &= (-kA'(v'^tv - v^tv'), -k\tilde{L}, k^t(A'(v'^tw - v^tw'))), \end{aligned}$$

where $\tilde{L}$ is defined as above.

We can combine $\mathfrak{so}(p,q)$ with Z^* to give $\mathfrak{so}(p,q+1)$ via

$$(K,v) \mapsto \begin{pmatrix} K & -k^{1/2}A'v \\ -k^{1/2}\,{}^t v & 0 \end{pmatrix}.$$

The subset $\mathfrak{r} = \{(0,L,M,u,0,w) \in \mathfrak{g}\}$ is a 2-step nilpotent ideal of $\mathfrak{g}$ (abelian when $r=n$, i.e., when the rank of the Ricci tensor is half the dimension of the manifold). Hence, when $p+q=r>1$, $\mathfrak{r}$ is the radical of $\mathfrak{g}$ and the semisimple Levi factor of $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(p,q+1)$.

4. Some corollaries

Corollary 6 *Let (M_i,ω_i,s_i), $i=1,2$ be symmetric symplectic spaces of the same dimension $2n$ with $W_i=0$ with semisimple transvection groups G_i. Then $G_1^{\mathbb{C}} = G_2^{\mathbb{C}}$.*

Proof. $SL(n+1,\mathbb{R})$ and $SU(p+1,q)$ both have $SL(n+1,\mathbb{C})$ as complexification. ■

Corollary 7 *Let (M,ω,s) be a compact, simply-connected symmetric symplectic space of dimension $2n$ such that $W=0$ then (M,ω,s) is $\mathbb{P}_n(\mathbb{C})$.*

Proof. This follows immediately from the list in Theorem 2. The only case where G/K is compact is when $G=SU(n+1)$ and $K=U(n)$. ■

In dimension 4 we have the following list of possibilities (up to coverings) for M:

- $SL(3,\mathbb{R})/GL(2,\mathbb{R})$;
- $SU(1,2)/U(2)$;
- $SU(2,1)/U(1,1)$;
- $SU(3)/U(2)$;
- $\lambda=0$ cases corresponding to:
    - $\mathrm{Rank}\,A=1$, $p=0$ or $p=1$;
    - $\mathrm{Rank}\,A=2$, $p=0$, $p=1$ or $p=2$.

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Drinfeld realization of quantum superalgebras and $U_q(\hat{\mathfrak{osp}}(1,2))$

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Abstract. In this paper, we extend the generalization of Drinfeld realization of quantum affine algebras to quantum affine superalgebras with its Drinfeld comultiplication and its Hopf algebra structure, which depends on a function $g(z)$ satisfying the relation: $g(z) = g(z^{-1})^{-1}$. In particular, we present the Drinfeld realization of $U_q(\hat{\mathfrak{osp}}(1,2))$ and its Serre relations.

Keywords: quantum superalgebra, Drinfeld realization, Serre relation

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Dedicated to our friend Moshé Flato

1. Introduction

Quantum groups as a noncommutative and noncocommutative Hopf algebras were discovered by Drinfeld [8] and Jimbo [16]. The standard definition of a quantum group is given as a deformation of universal enveloping algebra of a simple (super-) Lie algebra by the basic generators and the relations based on the data coming from the corresponding Cartan matrix. However, for the case of quantum affine algebras, there is a different aspect of the theory, namely their loop realizations. The first approach was given by Faddeev, Reshetikhin, and Takhtajan [12] and Reshetikhin and Semenov-Tian-Shansky [17], who obtained a realization of the quantum loop algebra $U_q(\mathfrak{g} \otimes \mathbb{C}[t,t^{-1}])$ via a canonical solution of the Yang-Baxter equation depending on a parameter $z \in \mathbb{C}$. On the other hand, Drinfeld [9] gave another realization of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ and its special degeneration called the Yangian, which is widely used in constructions of special representation of affine quantum algebras [14]. In [9], Drinfeld only gave the realization of the quantum affine algebras as an algebra, and as an algebra this realization is equivalent to the approach above [3] through certain Gauss decomposition for the case of $U_q(\hat{\mathfrak{gl}}(n))$. Certainly, the most important aspect of the structures of the quantum groups is its Hopf algebra structure, especially its comultiplication. Drinfeld also constructed a new Hopf algebra structure for this loop realization. The new comultiplication

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in this formulation, which we call the Drinfeld comultiplication, is simple and has very important applications [7, 5].

In [4], we observe that in the Drinfeld realization of quantum affine algebras $U_q(\hat{\mathfrak{sl}}(n))$, the structure constants are certain rational functions $g_{ij}(z)$, whose functional property of $g_{ij}(z)$ decides completely the Hopf algebra structure. In particular, for the case of $U_q(\hat{\mathfrak{sl}}_2)$, its Drinfeld realization is given completely in terms of a function $g(z)$, which has the following function property: $g(z) = g(z^{-1})^{-1}$. This leads us to generalize this type of Hopf algebras. Namely, we can substitute $g_{ij}(z)$ by other functions that satisfy the functional property of $g_{ij}(z)$, to derive new Hopf algebras.

In this paper, we will further extend the generalization of the Drinfeld realization of $U_q(\hat{\mathfrak{sl}}_2)$ to derive quantum affine superalgebras. As an example, we will also present the quantum affine superalgebra $U_q(\hat{\mathfrak{osp}}(1,2))$ in terms of the new formulation, in particular, we present the Serre relations in terms of the current operators.

The paper is organized as the following: in Section 2, we recall the main results in [4] about the generalization of Drinfeld realization of $U_q(\hat{\mathfrak{sl}}(2))$; in Section 3, we present the definition of the generalized Drinfeld realization of quantum superalgebras; in Section 4, we present the formulation of $U_q(\hat{\mathfrak{osp}}(1,2))$.

2. Drinfeld realization of quantum algebras

In [4], we derive a generalization of Drinfeld realization of $U_q(\hat{\mathfrak{sl}}_n)$. For the case of $U_q(\mathfrak{sl}_2)$, we first present the complete definition.

Let $g(z)$ be an analytic functions satisfying the following property that $g(z) = g(z^{-1})^{-1}$ and $\delta(z)$ be the distribution with support at 1.

Definition 2.1 *$U_q(g, f\mathfrak{sl}_2)$ is an associative algebra with unit 1 and the generators: $x^{\pm}(z)$, $\varphi(z)$, $\psi(z)$, a central element c and a nonzero complex parameter q, where $z \in \mathbb{C}^*$. $\varphi(z)$ and $\psi(z)$ are invertible. In terms of the generating functions: the defining relations are*

$$\varphi(z)\varphi(w) = \varphi(w)\varphi(z),$$

$$\psi(z)\psi(w) = \psi(w)\psi(z),$$

$$\varphi(z)\psi(w)\varphi(z)^{-1}\psi(w)^{-1} = \frac{g(z/wq^{-c})}{g(z/wq^{c})},$$

$$\varphi(z)x^{\pm}(w)\varphi(z)^{-1} = g(z/wq^{\mp\frac{1}{2}c})^{\pm 1}x^{\pm}(w),$$

$$\psi(z)x^{\pm}(w)\psi(z)^{-1} = g(w/zq^{\mp\frac{1}{2}c})^{\mp 1}x^{\pm}(w),$$

$$[x^{+}(z), x^{-}(w)] = \frac{1}{q-q^{-1}}\left(\delta(\frac{z}{w}q^{-c})\psi(wq^{\frac{1}{2}c}) - \delta(\frac{z}{w}q^{c})\varphi(zq^{\frac{1}{2}c})\right),$$

$$x^{\pm}(z)x^{\pm}(w) = g(z/w)^{\pm 1}x^{\pm}(w)x^{\pm}(z).$$

Theorem 2.1 *The algebra* $U_q(g, f\mathfrak{sl}_2)$ *has a Hopf algebra structure, which are given by the following formulas.*

Coproduct: Δ

$$
\begin{aligned}
&(0)\quad \Delta(q^c) = q^c \otimes q^c,\\
&(1)\quad \Delta(x^+(z)) = x^+(z)\otimes 1 + \varphi(zq^{\frac{c_1}{2}})\otimes x^+(zq^{c_1}),\\
&(2)\quad \Delta(x^-(z)) = 1\otimes x^-(z) + x^-(zq^{c_2})\otimes \psi(zq^{\frac{c_2}{2}}),\\
&(3)\quad \Delta(\varphi(z)) = \varphi(zq^{-\frac{c_2}{2}})\otimes \varphi(zq^{\frac{c_1}{2}}),\\
&(4)\quad \Delta(\psi(z)) = \psi(zq^{\frac{c_2}{2}})\otimes \psi(zq^{-\frac{c_1}{2}}),
\end{aligned}
$$

where $c_1 = c\otimes 1$ *and* $c_2 = 1\otimes c$.

Counit: ε

$$
\varepsilon(q^c) = 1 \quad \varepsilon(\varphi(z)) = \varepsilon(\psi(z)) = 1,
$$
$$
\varepsilon(x^{\pm}(z)) = 0.
$$

Antipode: *a*

$$
\begin{aligned}
&(0)\quad a(q^c) = q^{-c},\\
&(1)\quad a(x^+(z)) = -\varphi(zq^{-\frac{c}{2}})^{-1}x^+(zq^{-c}),\\
&(2)\quad a(x^-(z)) = -x^-(zq^{-c})\psi(zq^{-\frac{c}{2}})^{-1},\\
&(3)\quad a(\varphi(z)) = \varphi(z)^{-1},\\
&(4)\quad a(\psi(z)) = \psi(z)^{-1}.
\end{aligned}
$$

Strictly speaking, $U_q(g, f\mathfrak{sl}_2)$ is not an algebra. This concept, which we call a functional algebra, has already been used before [18], etc.

The Drinfeld realization for the case of $U_q(\hat{\mathfrak{sl}}_2)$ [9] as a Hopf algebra is different, and it an algebra and Hopf algebra defined with current operators in terms of formal power series.

Let $g(z)$ be an analytic functions that satisfying the following property that $g(z) = g(z^{-1})^{-1} = G^+(z)/G^-(z)$, where $G^{\pm}(z)$ is an analytic function without poles except at 0 or ∞ and $G^{\pm}(z)$ have no common zero point. Let $\delta(z) = \sum_{n\in\mathbb{Z}} z^n$, where z is a formal variable.

Definition 2.2 *The algebra* $U_q(g, \mathfrak{sl}_2)$ *is an associative algebra with unit 1 and the generators:* $\bar{a}(l)$, $\bar{b}(l)$, $x^{\pm}(l)$, *for* $l\in\mathbb{Z}$ *and a central element c. Let z be a formal variable and*

$$
x^{\pm}(z) = \sum_{l\in\mathbb{Z}} x^{\pm}(l)z^{-l},
$$

$$
\varphi(z) = \sum_{m\in\mathbb{Z}} \varphi(m)z^{-m} = \exp[\sum_{m\in\mathbb{Z}_{\le 0}} \bar{a}(m)z^{-m}]\exp[\sum_{m\in\mathbb{Z}_{>0}} \bar{a}(m)z^{-m}]
$$

and

$$\psi(z) = \sum_{m\in\mathbb{Z}} \psi(m)z^{-m} = \exp[\sum_{m\in\mathbb{Z}_{\leq 0}} \bar{b}(m)z^{-m}]\exp[\sum_{m\in\mathbb{Z}_{>0}} \bar{b}(m)z^{-m}].$$

In terms of the formal variables z, w, the defining relations are

$$\begin{aligned}
&a(l)a(m) = a(m)a(l),\\
&b(l)b(m) = b(m)b(l),\\
&\varphi(z)\psi(w)\varphi(z)^{-1}\psi(w)^{-1} = \frac{g(z/wq^{-c})}{g(z/wq^{c})},\\
&\varphi(z)x^{\pm}(w)\varphi(z)^{-1} = g(z/wq^{\mp\frac{1}{2}c})^{\pm 1}x^{\pm}(w),\\
&\psi(z)x^{\pm}(w)\psi(z)^{-1} = g(w/zq^{\mp\frac{1}{2}c})^{\mp 1}x^{\pm}(w),\\
&[x^{+}(z),x^{-}(w)] = \frac{1}{q-q^{-1}}\left(\delta(\frac{z}{w}q^{-c})\psi(wq^{\frac{1}{2}c}) - \delta(\frac{z}{w}q^{c})\varphi(zq^{\frac{1}{2}c})\right),\\
&G^{\mp}(z/w)x^{\pm}(z)x^{\pm}(w) = G^{\pm}(z/w)x^{\pm}(w)x^{\pm}(z),
\end{aligned}$$

where by $g(z)$ *we mean the Laurent expansion of* $g(z)$ *in a region* $r_1 > |z| > r_2$.

Theorem 2.2 *The algebra* $U_q(g,\mathfrak{sl}_2)$ *has a Hopf algebra structure. The formulas for the coproduct* Δ*, the counit* ε *and the antipode* a *are the same as given in Theorem 2.1.*

Here, one has to be careful with the expansion of the structure functions $g(z)$ and $\delta(z)$, for the reason that the relations between $x^{\pm}(z)$ and $x^{\pm}(z)$ are different from the case of the functional algebra above.

Example 2.1 Let $\bar{g}(z)$ be a an analytic function such that $\bar{g}(z^{-1}) = -z^{-1}\bar{g}(z)$. Let $g(z) = q^{-2}\frac{\bar{g}(q^2 z)}{\bar{g}(q^{-2}z)}$. Then $g(z) = g(z^{-1})^{-1}$. With this $g(z)$, we can define an algebra $U_q(g,\mathfrak{sl}_2)$ by Definition 2.2.

Case I. Let $\bar{g}(z)$=$1-z$. $\varphi(m) = \psi(-m) = 0$ for $m \in \mathbb{Z}_{>0}$ and $\varphi(0)\psi(0) = 1$. Then this algebra is $U_q(\hat{\mathfrak{sl}}_2)$.

Case II. Let $\bar{g}(z) = \theta_p(z) = \prod_{j>0}(1-p^j)(1-p^{j-1}z)(1-p^j z^{-1})$ be the Jacobi's theta function. $\theta_p(z^{-1}) = -z^{-1}\theta_p(z) = \theta_p(pz)$. We will call this algebra $U_q(\theta,\mathfrak{sl}_2)$. We take the expansion of $g(z)$ in the region $|q^2| > |z| > |q^2 p|$. If we take the limit that p goes to zero, we would recover $U_q(\hat{\mathfrak{sl}}_2)$.

3. Drinfeld realization of quantum superalgebras

Similarly, we can use the same idea to define quantum affine superalgebras, which is an extension of the generalization in the section above. Again, we will start from the same function $g(z)$, namely, let $g(z)$ be an analytic functions satisfying the following property that $g(z) = g(z^{-1})^{-1}$ and $\delta(z)$ be the distribution with support at 1.

Definition 3.1 *$U_q(g, f\mathfrak{s})$ is an a $\mathbb{Z}_2$-graded associative algebra with unit 1 and the generators: $x^{\pm}(z)$, $\varphi(z)$, $\psi(z)$, a central element c and a nonzero complex parameter q, where $z \in \mathbb{C}^*$, $x^{\pm}(z)$ are graded 1 (mod 2), and $\varphi(z)$, $\psi(z)$ and c are graded 0 (mod 2). $\varphi(z)$ and $\psi(z)$ are invertible. In terms of the generating functions: the defining relations are*

$$\begin{aligned}
&\varphi(z)\varphi(w) = \varphi(w)\varphi(z),\\
&\psi(z)\psi(w) = \psi(w)\psi(z),\\
&\varphi(z)\psi(w)\varphi(z)^{-1}\psi(w)^{-1} = \frac{g(z/wq^{-c})}{g(z/wq^{c})},\\
&\varphi(z)x^{\pm}(w)\varphi(z)^{-1} = g(z/wq^{\mp\frac{1}{2}c})^{\pm 1}x^{\pm}(w),\\
&\psi(z)x^{\pm}(w)\psi(z)^{-1} = g(w/zq^{\mp\frac{1}{2}c})^{\mp 1}x^{\pm}(w),\\
&\{x^{+}(z), x^{-}(w)\} = \frac{1}{q-q^{-1}}\left(\delta(\frac{z}{w}q^{-c})\psi(wq^{\frac{1}{2}c}) - \delta(\frac{z}{w}q^{c})\varphi(zq^{\frac{1}{2}c})\right),\\
&x^{\pm}(z)x^{\pm}(w) = -g(z/w)^{\pm 1}x^{\pm}(w)x^{\pm}(z),
\end{aligned}$$

where $\{x, y\} = xy + yx$.

Here we remark that the above relations are basically the same as in that of Definition 2.1 except the relation between $x^{\pm}(z)$ and $x^{\pm}(w)$ respectively, which differs by a negative sign.

Accordingly we have that, for the tensor algebra, the multiplication is defined for homogeneous elements a, b, c, d by $(a\otimes b)(c\otimes d) = (-1)^{[b][c]}(ac\otimes bd)$, where $[a] \in \mathbb{Z}_2$ denotes the grading of the element a.

Similarly we have:

Theorem 3.1 *The algebra $U_q(g, f\mathfrak{s})$ has a graded Hopf algebra structure, whose coproduct, counit and antipode are given by the same formulas of $U_q(q, f\mathfrak{sl}_2)$ in Theorem 2.1.*

As for the case of $U_q(g, f\mathfrak{sl}_2)$ is not a graded algebra but rather a graded functional algebra.

Let $g(z) = g(z^{-1})^{-1} = G^{+}(z)/G^{-}(z)$, where $G^{\pm}(z)$ is an analytic function without poles except at 0 or ∞ and $G^{\pm}(z)$ have no common zero point.

Definition 3.2 *The algebra $U_q(g, \mathfrak{s})$ is $\mathbb{Z}_2$ graded associative algebra with unit 1 and the generators: $\bar{a}(l)$, $\bar{b}(l)$, $x^{\pm}(l)$, for $l \in \mathbb{Z}$ and a central element c, where $x^{\pm}(l)$ are graded 1 (mod 2) and the rest are graded 0 (mod 2). Let z be a formal variable and*

$$x^{\pm}(z) = \sum_{l\in\mathbb{Z}} x^{\pm}(l)z^{-l},$$

$$\varphi(z)=\sum_{m\in\mathbb{Z}}\varphi(m)z^{-m}=\exp[\sum_{m\in\mathbb{Z}_{\leq 0}}\bar{a}(m)z^{-m}]\exp[\sum_{m\in\mathbb{Z}_{>0}}\bar{a}(m)z^{-m}]$$

and

$$\psi(z)=\sum_{m\in\mathbb{Z}}\psi(m)z^{-m}=\exp[\sum_{m\in\mathbb{Z}_{\leq 0}}\bar{b}(m)z^{-m}]\exp[\sum_{m\in\mathbb{Z}_{>0}}\bar{b}(m)z^{-m}].$$

In terms of the formal variables z, w, the defining relations are

$$\begin{aligned}
&\varphi(z)\varphi(w)=\varphi(w)\varphi(z),\\
&\psi(z)\psi(w)=\psi(w)\psi(z),\\
&\varphi(z)\psi(w)\varphi(z)^{-1}\psi(w)^{-1}=\frac{g(z/wq^{-c})}{g(z/wq^{c})},\\
&\varphi(z)x^{\pm}(w)\varphi(z)^{-1}=g(z/wq^{\mp\frac{1}{2}c})^{\pm 1}x^{\pm}(w),\\
&\psi(z)x^{\pm}(w)\psi(z)^{-1}=g(w/zq^{\mp\frac{1}{2}c})^{\mp 1}x^{\pm}(w),\\
&\{x^{+}(z),x^{-}(w)\}=\frac{1}{q-q^{-1}}\left(\delta(\frac{z}{w}q^{-c})\psi(wq^{\frac{1}{2}c})-\delta(\frac{z}{w}q^{c})\varphi(zq^{\frac{1}{2}c})\right),\\
&(G^{\mp}(z/w))x^{\pm}(z)x^{\pm}(w)=-(G^{\pm}(z/w))x^{\pm}(w)x^{\pm}(z),
\end{aligned}$$

where by $g(z)$ we mean the Laurent expansion of $g(z)$ in a region $r_1>|z|>r_2$.

The above relations are basically the same as in that of Definition 2.2 except the relation between $x^{\pm}(z)$ and $x^{\pm}(w)$ respectively, which differs by a negative sign. The expansion direction of the structure functions $g(z)$ and $\delta(z)$ is very important, for the reason that the relations between $x^{\pm}(z)$ and $x^{\pm}(z)$ are different from the case of the functional algebra above.

Theorem 3.2 *The algebra $U_q(g,\mathfrak{s})$ has a Hopf algebra structure. The formulas for the coproduct Δ, the counit ε and the antipode a are the same as given in Theorem 2.1.*

Example 3.1 *Let $\bar{g}(z)=1$. From [1, 19], we can see that $U_q(1,\mathfrak{s})$ is basically the same as $U_q(\hat{\mathfrak{gl}}(1,1))$.*

4. $U_q(o\hat{s}p(1,2))$ and its Serre relations

For a rational function $g(z)$ that satisfies $g(z)=g(z^{-1})^{-1}$, it is clear that $g(z)$ is determined by its poles and its zeros, which are paired to satisfy the relations above. For the simplest case (except $g(z)=1$) that $g(z)$ has only one pole and one zero, we have

$$g(z)=\frac{zp-1}{z-p},$$

where p is the location of the pole of $g(z)$. Unfortunately, in this case, we do not know anyway to identify the algebra $U_q(g,\mathfrak{s})$ with other know structures.

Then, comes the second simplest case that $g(z)$ has two poles and two zeros. We know also that $g(z)$ must be in the form that

$$g(z) = \frac{zp_1 - 1}{z - p_1}\frac{zp_2 - 1}{z - p_2}.$$

In this section, we shall establish that for this case, $U_q(g,\mathfrak{s})$ is related to the affine quantum superalgebra $U_q(\hat{\mathfrak{osp}}(1,2))$.

From now on, let us fix $g(z)$ to be $\frac{zp_1-1}{z-p_1}\frac{zp_2-1}{z-p_2}$.

As in [7, 6], for the case of quantum affine algebras, it is very important to understand the poles and zero of the product of current operators. We will start with the relations between $X^+(z)$ with itself. From the definition, we know that

$$(z-p_1w)(z-p_2w)X^+(z)X^+(w) = -(zp_1-w)(zp_2-w)X^+(w)X^+(z).$$

From this, we know that $X^+(z)X^+(w)$ has two poles, which are located at $(z-p_1w)=0$ and $(z-p_2w)=0$.

This also implies that

Proposition 4.1 $X^+(z)X^+(w) = 0$, *when* $z = w$.

If we assume that $U_q(g,\mathfrak{s})$ is related to some quantized affine superalgebra, then we can see that the best chance we have is $U_q(\hat{\mathfrak{osp}}(1,2))$ by looking at the number of zeros and poles of $X^+(z)X^+(w)$.

However, for the case of $U_q(\hat{\mathfrak{osp}}(1,2))$, we know we need an extra Serre relation. For this, we will follow the idea of [13].

Let

$$\begin{aligned}
f(z_1,z_2) =&(z_1-p_1z_2)(z_1-p_2z_2),\\
Y^+(z,w) =&\frac{(z-p_1w)(z-p_2w)}{z-w}X^+(z)X^+(w),\\
Y^+(z_1,z_2,z_3) =&\frac{(z_1-p_1z_2)(z_1-p_2z_2)}{z_1-z_2}\frac{(z_1-p_1z_3)(z_1-p_2z_3)}{z_1-z_3}\times\\
&\frac{(z_2-p_1z_3)(z_2-p_2z_3)}{z_2-z_3}X^+(z_1)X^+(z_2)X^+(z_3),\\
F(z_1,z_2,z_3) =&(z_1-p_1z_2)(z_1-p_2z_2)(z_3-p_1z_1)(z_3-p_2z_1)(z_2-p_1z_3)(z_2-p_2z_3)\\
=&f(z_1,z_2)f(z_2,z_3)f(z_3,z_1),\\
\bar{F}(z_1,z_2,z_3) =&f(z_2,z_1)f(z_2,z_3)f(z_3,z_1).
\end{aligned}$$

Let $V_(z_1,z_2,z_3)$ be the algebraic variety of the zeros of $F(z_1,z_2,z_3)$. Let $V(a(z_1),a(z_2),a(z_3))$, be the image of the action of a on this variety, where $a \in$

S_3, the permutation group on z_1, z_2, z_3. Let $\bar{V}_{(}z_1, z_2, z_3)$ be the algebraic variety of the zeros of $\bar{F}(z_1, z_2, z_3)$. Let $\bar{V}(a(z_1), a(z_2), a(z_3))$, be the image of the action of a on this variety, where $a \in S_3$.

Proposition 4.2 *$Y^+(z, w)$ has no poles and is symmetry with respect to z and w. $Y^+(z_1, z_2, z_3)$ has no poles and is symmetry with respect to z_1, z_2, z_3.*

Following the idea in [13], we would like to define the following conditions that may be imposed on our algebra.

Zero Condition I: $Y^+(z_1, z_2, z_3)$ is zero on at least one line that crosses (0,0,0), and this line must lie in a $V(a(z_1), a(z_2), a(z_3))$ for some element $a \in S_3$.

Zero Condition II: $Y^+(z_1, z_2, z_3)$ is zero on at least one line that crosses (0,0,0), and this line must lie in a $\bar{V}(a(z_1), a(z_2), a(z_3))$ for some element $a \in S_3$.

For the line that crosses $(0, 0, 0)$ where $Y^+(z_1, z_2, z_3)$ is zero, we call it the zero line of $Y^+(z_1, z_2, z_3)$. Because $Y^+(z_1, z_2, z_3)$ is symmetric with respect to the action of S_3 on z_1, z_2, z_3, if a line is the zero line of $Y^+(z_1, z_2, z_3)$, then clearly the orbit of the line under the action of S_3 is also an zero line.

Remark 1 There is a simple symmetry that we would prefer to choose the function $F(z_1, z_2, z_3)$ to determine the variety $V(z_1, z_2, z_3)$. We have that

$$F(z_1, z_2, z_3) = f(z_1, z_2) f(z_2, z_3) f(z_3, z_1).$$

Let S_2^1 be the permutation group acting on z_1, z_2. Let S_2^2 be the permutation group acting on z_2, z_3. Let S_2^1 be the permutation group acting on z_3, z_1. Clearly, [13] we can choose from a family of varieties determined by the functions

$$f(a_1(z_1), a_1(z_2)) f(a_2(z_2), a_2(z_3)) f(a_3(z_3), a_3(z_1))$$

for $a_1 \in S_2^1$, $a_2 \in S_2^2$, $a_3 \in S_2^3$. For each such a function

$$f(a_1(z_1), a_1(z_2)) f(a_2(z_2), a_2(z_3)) f(a_3(z_3), a_3(z_1)),$$

we can attach a oriented diagram, whose nods are z_1, z_2, z_3, and the arrows are given by $(a_1(z_1) \to a_1(z_2))$, $(a_2(z_2) \to a_2(z_3))$ $(a_3(z_3) \to a_3(z_1))$. For example the diagram of $F(z_1, z_2, z_3)$ is given by

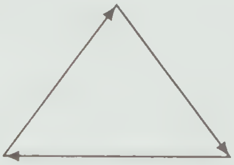

Diagram I

While $f(z_2, z_1) f(z_2, z_3) f(z_3, z_1)$ is denoted by

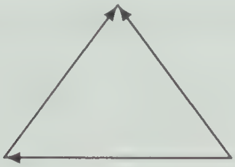

Diagram II

It is not difficult to see that the diagram for $F(z_1,z_2,z_3)$ is symmetric in the sense that all the points are equivalent, but for the second situation, the top point z_1 is different from the other two, in the sense that there are two arrows coming to z_1, one to z_3 and none to z_2. There is only one other such a diagram given by $F(z_1,z_3,z_2)$, which however comes from the S_3 action on $F(z_1,z_2,z_3)$.

From the above, we have the following:

Proposition 4.3 *Under the action of S_3 on the the family of varieties determined by the functions $f(a_1(z_1),a_1(z_2))f(a_2(z_2),a_2(z_3))f(a_3(z_3),a_3(z_1))$ for $a_1 \in S_2^1$, $a_2 \in S_2^2$, $a_3 \in S_2^3$, there are two orbits. One of the orbit consists of the two varieties determined by $F(z_1,z_2,z_3)$ and $F(z_1,z_2,z_3)$; and the rest forms another orbit.*

This shows that indeed we have two choices with respect the zero conditions: the Zero condition I and the Zero condition II.

We know that a zero line is always in the form

$$z_1 = q_1 z_2 = q_2 z_3.$$

Then we have

Proposition 4.4 *If we impose the Zero condition I on the algebra $U_q(g,\mathfrak{s})$, we have*

$$p_1 = p_2^{-2} \quad or \quad p_2 = p_1^{-2}.$$

Proof. The proof is very simple. Because of the action of S_3, we know that one of the line must lie in $V(z_1,z_2,z_3)$. Let us assume this line to be $z_1 = z_2 q_1 = q_2 z_3$.

We know immediately that q_1 must be p_1 or p_2.

Let us first deal with the case where $q_1 = p_1$. We also know that q_2 must be either p_1^{-1} or p_2^{-1} by looking at the relations between z_1 and z_3.

Case 1. Let $q_2 = p_1^{-1}$, which implies that p_1^{-2} must be either p_1 or p_2. Clearly, it cannot be p_1, which implies that the impossible condition $p_1 = 1$

Therefore, we have that $p_1^{-2} = p_2$, which is what we want.

Case 2. Let $q_2 = p_2^{-1}$, which implies that $p_1 p_2$ must be either p_1 or p_2. Clearly, it cannot be p_1 because it implies $p_2 = 1$, it cannot be neither be p_2, which implies that $p_1 = 1$. This completes the proof for $p_2 = p_1^{-2}$.

Similarly, if we have that $q_1 = p_2$, we can then show that $p_1 = p_2^{-2}$.

However from the algebraic point of view, the two condition are equivalent in the sense that p_1 and p_2 are symmetric.

Also we have that

Proposition 4.5 *If we impose the Zero condition II on the algebra* $U_q(g,\mathfrak{s})$, *we have*

$$p_1 = p_2^2 \quad or \quad p_2 = p_1^2.$$

However we also have that: **If we impose the Zero condition II on the algebra $U_q(g,\mathfrak{s})$, then $U_q(g,\mathfrak{s})$ is not a Hopf algebra anymore.**

The reason is that the Zero condition II cannot be satisfied by comultiplication, which can be checked by direct calculation.

This is the most important reason that we will choose the Zero condition I to be imposed on the algebra $U_q(g,\mathfrak{s})$, which comes actually from the consideration of Hopf algebra structure. Namely, if we choose $V(z_1,z_2,z_3)$ or the equivalent ones which has the same diagram presentation as Diagram I to define the zero line of $Y^+(z_1,z_2,z_3)$, then, the quotient algebra derived from the **Zero Condition I** is still a Hopf algebra with the same Hopf algebra structure (comultiplication, counit and antipode).

From now on, we impose the Zero condition I on the algebra $U_q(g,\mathfrak{s})$, and let us fix the notation such that

$$p_1 = q^2, \qquad p_2 = q^{-1}.$$

Similarly, we define

$$Y^-(z,w) = \frac{(z-p_1^{-1}w)(z-p_2^{-1}w)}{z-w} X^+(z)X^+(w),$$

$$Y^-(z_1,z_2,z_3) = \frac{(z_1-p_1^{-1}z_2)(z_1-p_2^{-1}z_2)}{z_1-z_2}\,\frac{(z_1-p_1^{-1}z_3)(z_1-p_2^{-1}z_3)}{z_1-z_3}\times$$

$$\frac{(z_2-p_1^{-1}z_3)(z_2-p_2^{-1}z_3)}{z_2-z_3} X^+(z_1)X^+(z_2)X^+(z_3),$$

We now define the q-Serre relation.

q-Serre relations

$Y^+(z_1,z_2,z_3)$ is zero on the line $z_1 = z_2q^{-1} = z_3q^{-2}$.

$Y^-(z_1,z_2,z_3)$ is zero on the line $z_1 = z_2q = z_3q^2$.

The q-Serre relations can also be formulated in more algebraic way.

Proposition 4.6 *The q-Serre relations are equivalent to the following two relations:*

$$\begin{aligned}
&\frac{(z_3-z_1q^{-1})(z_3-z_1q^3)(z_1-z_2q^2)}{z_3-z_1q}X^+(z_3)X^+(z_1)X^+(z_2)+\\
&\frac{(z_2-z_1q^2)(z_2-z_1q^{-1})(z_3-z_1q^{-1})(z_3-z_1q^3)}{(z_1-z_2q^{-1})(z_3-z_1q)}X^+(z_3)X^+(z_2)X^+(z_1))-\\
&\frac{((z_1-z_3q^2)(z_1-z_3q)(z_1q-z_3q^{-1})(z_1-z_2q^2)}{(z_1-z_3)(z_3-z_1q^2)}X^+(z_1)X^+(z_2)X^+(z_3)-\\
&\frac{((z_1-z_3q^2)(z_1-z_3q)(z_1q-z_3q^{-1})(z_2-z_1q^2)(z_2-z_1q^{-1})}{(z_1-z_3)(z_3-z_1q^2)(z_1-z_2q^{-1})}\times\\
&\qquad\qquad\times X^+(z_2)X^+(z_1)X^+(z_3))=0,
\end{aligned}$$

$$\begin{aligned}
&\frac{(z_3-z_1q)(z_3-z_1q^{-3})(z_1-z_2q^{-2})}{z_3-z_1q^{-1}}X^-(z_3)X^-(z_1)X^-(z_2)+\\
&\frac{(z_2-z_1q^{-2})(z_2-z_1q)(z_3-z_1q)(z_3-z_1q^{-3})}{(z_1-z_2q)(z_3-z_1q^{-1})}X^-(z_3)X^-(z_2)X^-(z_1))-\\
&\frac{((z_1-z_3q^{-2})(z_1-z_3q^{-1})(z_1q-z_3q)(z_1-z_2q^{-2})}{(z_1-z_3)(z_3-z_1q^{-2})}X^-(z_1)X^-(z_2)X^-(z_3)-\\
&\frac{((z_1-z_3q^{-2})(z_1-z_3q^{-1})(z_1q-z_3q)(z_2-z_1q^{-2})(z_2-z_1q)}{(z_1-z_3)(z_3-z_1q^{-2})(z_1-z_2q)}\times\\
&\qquad\qquad\times X^-(z_2)X^-(z_1)X^-(z_3))=0,
\end{aligned}$$

where the coefficient functions of the relations above are expanded in the region the expansion region of the corresponding monomial of the product of $X^{\pm}(z_j)$.

The proof is a simple calculation. (See also [11].) It is not very difficult to show that this relation will give us the classical Serre relations, but unfortunately, we still do not know how to write down a simple Serre relation like that of $U_q(\hat{\mathfrak{sl}}(3))$.

Definition 4.1 $U_q(g,\bar{\mathfrak{s}})$ *is the quotient algebra of* $U_q(g,\mathfrak{s})$ *with the kernel defined by the q-Serre relations,* $\bar{a}(m)=0$, $m>0$, $\bar{b}(m)$, $m<0$ *and* $a(0)=-b(0)$, *and* $g(z)$ *is expanded around 0.*

Theorem 4.7 $U_q(g,\bar{\mathfrak{s}})$ *is also a Hopf algebra, whose Hopf algebra structure is the same as that of* $U_q(g,\mathfrak{s})$.

This can be proven by calculation as in [4]

Another immediately result we can derive is that $U_q(g,\mathfrak{s})$ [2, 15] can be identified with $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$, where $U_q(\mathfrak{o}\hat{\mathfrak{s}}\mathfrak{p}(1,2))$ is derived from the FRTS realization using R-matrix and L-operators.

Theorem 4.8 *$U_q(g,\mathfrak{s})$ with the q-Serre relation is isomorphic to $U_q(\hat{\mathfrak{osp}}(1,2)$.*

It is still an open and interesting question to study $U_q(g,\mathfrak{s})$ given by other function $g(z)$.

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Quantum homogeneous spaces and quasi-Hopf algebras

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Abstract. We propose a formulation of the quantization problem of Manin quadruples, and show that a solution to this problem yields a quantization of the corresponding Poisson homogeneous spaces. We then solve both quantization problems in an example related to quantum spheres.

Keywords: Manin triples, Manin pairs, Poisson-Lie groups, homogeneous spaces, Hopf algebras, quasi-Hopf algebras

Mathematics Subject Classifications (2000): 16B62, 53D17, 17B37, 20G42

To the memory of Moshé Flato

Introduction

According to a theorem of Drinfeld, formal Poisson homogeneous spaces over a formal Poisson-Lie group G_+ with Lie algebra $\mathfrak{g}_+$ correspond bijectively to G_+-conjugation classes of Lagrangian (i.e., maximal isotropic) Lie subalgebras $\mathfrak{h}$ of the double Lie algebra $\mathfrak{g}$ of $\mathfrak{g}_+$. The formal Poisson homogenous space is then $G_+/(G_+ \cap H)$, where H is the formal Lie group with Lie algebra $\mathfrak{h}$. The corresponding quantization problem is to deform the algebra of functions over the homogeneous space to an algebra-module over the quantized enveloping algebra of $\mathfrak{g}_+$.

In this paper, we show that there is a Poisson homogeneous structure on the formal homogeneous space G/H, such that the embedding of $G_+/(G_+ \cap H)$ in G/H is Poisson, where G is the formal Lie group with Lie algebra $\mathfrak{g}$. It is therefore natural to seek a quantization of the function algebra of $G_+/(G_+ \cap H)$ as a quotient of a quantization of G/H.

The data of $(\mathfrak{g}, \mathfrak{h})$ and the r-matrix of $\mathfrak{g}$ constitute an example of a quasitriangular Manin pair (see Section 1.1). We introduce the notion of the quantization of a quasitriangular Manin pair, which consists of a quasitriangular Hopf algebra quantizing the Lie bialgebra $\mathfrak{g}$, quasi-Hopf algebras quantizing the Manin pair

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$(\mathfrak{g},\mathfrak{h})$, and a twist element relating both structures. We then show (Theorem 2.1) that this data gives rise to a quantization of G/H.

The quadruple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$ formed by adjoining $\mathfrak{h}$ to the Manin triple of $\mathfrak{g}_+$ is called a Manin quadruple. The quantization of a Manin quadruple is the additional data of Hopf algebras quantizing the Manin triple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-)$, subject to compatibility conditions with the quantization of the underlying quasitriangular Manin pair. We show that any quantization of a given quadruple gives rise to a quantization of the corresponding homogeneous space (Theorem 3.1).

Finally, in Section 4.1, we explicitly solve the problem of quantizing a Manin quadruple in a situation related to quantum spheres.

1. Manin quadruples and Poisson homogeneous spaces

In this section, we define Lie-algebraic structures, i.e., quasitriangular Manin pairs and Manin quadruples, and the Poisson homogeneous spaces naturally associated to them.

1.1. Quasitriangular Manin pairs

Recall that a *quasitriangular Lie bialgebra* is a triple $(\mathfrak{g},r,\langle\ ,\ \rangle_{\mathfrak{g}})$, where $\mathfrak{g}$ is a complex Lie algebra, $\langle\ ,\ \rangle_{\mathfrak{g}}$ is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, and r is an element of $\mathfrak{g}\otimes\mathfrak{g}$ such that $r+r^{(21)}$ is the symmetric element of $\mathfrak{g}\otimes\mathfrak{g}$ defined by $\langle\ ,\ \rangle_{\mathfrak{g}}$, and r satisfies the classical Yang-Baxter equation, $[r^{(12)},r^{(13)}]+[r^{(12)},r^{(23)}]+[r^{(13)},r^{(23)}]=0$. (Such Lie bialgebras are also called factorizable.)

Assume that $\mathfrak{g}$ is finite dimensional, and let G be a Lie group with Lie algebra $\mathfrak{g}$. Then G is equipped with the Poisson-Lie bivector $P_G=r^L-r^R$, where, for any element a of $\mathfrak{g}\otimes\mathfrak{g}$, we denote the right- and left-invariant 2-tensors on G corresponding to a by a^L and a^R. If $\mathfrak{g}$ is an arbitrary Lie algebra, the same statement holds for its formal Lie group.

We will call the pair $(\mathfrak{g},r,\langle\ ,\ \rangle_{\mathfrak{g}},\mathfrak{h})$ of a quasitriangular Lie bialgebra $(\mathfrak{g},r,\langle\ ,\ \rangle_{\mathfrak{g}})$ and a Lagrangian Lie subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ a *quasitriangular Manin pair*.

Assume that $(\mathfrak{g},r,\langle\ ,\ \rangle_{\mathfrak{g}},\mathfrak{h})$ is a quasitriangular Manin pair, and that L is a Lagrangian complement of $\mathfrak{h}$ in $\mathfrak{g}$. Let (ε^i) and (ε_i) be dual bases of $\mathfrak{h}$ and L, and set $r_{\mathfrak{h},L}=\sum_i\varepsilon^i\otimes\varepsilon_i$. The restriction to L of the Lie bracket of $\mathfrak{g}$ followed by the projection to the first factor in $\mathfrak{g}=\mathfrak{h}\oplus L$ yields an element $\varphi_{\mathfrak{h},L}$ of $\wedge^3\mathfrak{h}$. Let us set $f_{\mathfrak{h},L}=r_{\mathfrak{h},L}-r$; then $f_{\mathfrak{h},L}$ belongs to $\wedge^2\mathfrak{g}$.

Then the twist of the Lie bialgebra $(\mathfrak{g},\partial r)$ by $f_{\mathfrak{h},L}$ is the quasitriangular Lie quasi-bialgebra $(\mathfrak{g},\partial r_{\mathfrak{h},L},\varphi_{\mathfrak{h},L})$. The cocycle $\partial r_{\mathfrak{h},L}$ maps $\mathfrak{h}$ to $\wedge^2\mathfrak{h}$, so $(\mathfrak{h},(\partial r_{\mathfrak{h},L})_{|\mathfrak{h}},\varphi_{\mathfrak{h},L})$ is a sub-Lie quasi-bialgebra of $(\mathfrak{g},\partial r_{\mathfrak{h},L},\varphi_{\mathfrak{h},L})$.

1.2. Manin quadruples

1.2.1. *Definition*

A *Manin quadruple* is a quadruple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$, where $\mathfrak{g}$ is a complex Lie algebra, equipped with a nondegenerate invariant symmetric bilinear form $\langle\ ,\ \rangle_{\mathfrak{g}}$, and $\mathfrak{g}_+, \mathfrak{g}_-$ and $\mathfrak{h}$ are three Lagrangian subalgebras of $\mathfrak{g}$, such that $\mathfrak{g}$ is equal to the direct sum $\mathfrak{g}_+ \oplus \mathfrak{g}_-$ (see [11]).

In particular, $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. This implies that for any Lie group G with subgroups G_+, G_- integrating $\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-$, we obtain a Poisson-Lie group structure P_G on G, such that G_+ and G_- are Poisson-Lie subgroups of (G, P_G) (see [6]). We denote the corresponding Poisson structures on G_+ and G_- by P_{G_+} and P_{G_-}.

Any Manin quadruple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ gives rise to a quasitriangular Manin pair $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$, where $r = r_{\mathfrak{g}_+, \mathfrak{g}_-} = \sum_i e^i \otimes e_i$, and $(e^i), (e_i)$ are dual bases of $\mathfrak{g}_+$ and $\mathfrak{g}_-$.

1.2.2. *Examples*

In the case where $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is the Manin triple associated with the Sklyanin structure on a semisimple Lie group G_+, the Manin quadruples were classified in [12]. (See also [4].) It was shown in [14] that the Lagrangian subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ such that the intersection $\mathfrak{g}_+ \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_+$ correspond bijectively to the classical dynamical r-matrices for $\mathfrak{g}_+$. We will treat the quantization of this example in Section 4, in the case where $\mathfrak{g}_+ = \mathfrak{sl}_2$.

In [9], the following class of Manin quadruples was studied. Let $\bar{\mathfrak{g}}$ be a semisimple Lie algebra with nondegenerate invariant symmetric bilinear form $\langle\ ,\ \rangle_{\bar{\mathfrak{g}}}$, and Cartan decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_-$. Let $\mathscr{K}$ be a commutative ring equipped with a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle_{\mathscr{K}}$. Assume that $R \subset \mathscr{K}$ is a Lagrangian subring of $\mathscr{K}$, with Lagrangian complement Λ. Let us set $\mathfrak{g} = \bar{\mathfrak{g}} \otimes \mathscr{K}$, let us equip $\mathfrak{g}$ with the bilinear form $\langle\ ,\ \rangle_{\bar{\mathfrak{g}}} \otimes \langle\ ,\ \rangle_{\mathscr{K}}$ and let us set

$$\mathfrak{g}_+ = (\bar{\mathfrak{h}} \otimes R) \oplus (\bar{\mathfrak{n}}_+ \otimes \mathscr{K}), \quad \mathfrak{g}_- = (\bar{\mathfrak{h}} \otimes \Lambda) \oplus (\bar{\mathfrak{n}}_- \otimes \mathscr{K}), \quad \mathfrak{h} = \bar{\mathfrak{g}} \otimes R. \tag{1}$$

More generally, extensions of these Lie algebras, connected with the additional data of a derivation ∂ of $\mathscr{K}$ leaving $\langle\ ,\ \rangle_{\mathscr{K}}$ invariant and preserving R, were considered in [9]. Examples of quadruples $(\mathscr{K}, \partial, R, \langle\ ,\ \rangle_{\mathscr{K}})$, where $\mathscr{K}$ is an infinite-dimensional vector space, arise in the theory of complex curves.

1.3. Formal Poisson homogeneous spaces

In what follows, all homogeneous spaces will be formal, so if $\mathfrak{a}$ is a Lie algebra and A is the associated formal group, the function ring of A is $\mathscr{O}_A := (U\mathfrak{a})^*$, and if $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{a}$, and B is the associated formal group, the function ring of A/B is $\mathscr{O}_{A/B} := (U\mathfrak{a}/(U\mathfrak{a})\mathfrak{b})^*$.

We will need the following result on formal homogeneous spaces.

Lemma 1.1 *Let $\mathfrak{a}$ be a Lie algebra, let $\mathfrak{a}_+$ and $\mathfrak{b}$ be Lie subalgebras of $\mathfrak{a}$, and let A, A_+, and B be the associated formal groups. The restriction map*

$(U\mathfrak{a}/(U\mathfrak{a})\mathfrak{b})^* \to (U\mathfrak{a}_+/(U\mathfrak{a}_+(\mathfrak{a}_+ \cap \mathfrak{b})))^*$ is a surjective morphism of algebras from $\mathcal{O}_{A/B}$ to $\mathcal{O}_{A_+/(A_+\cap B)}$.

Proof. Let L_+ be a complement of $\mathfrak{a}_+ \cap \mathfrak{b}$ in $\mathfrak{a}_+$, and let L be a complement of $\mathfrak{b}$ in $\mathfrak{a}$, containing L_+. When V is a vector space, we denote by $S(V)$ its symmetric algebra. The following diagram is commutative

$$\begin{tikzcd} S(L_+) \arrow[r] \arrow[d] & U\mathfrak{a}_+/(U\mathfrak{a}_+(\mathfrak{a}_+ \cap \mathfrak{b})) \arrow[d] \\ S(L) \arrow[r] & U\mathfrak{a}/(U\mathfrak{a})\mathfrak{b} \end{tikzcd}$$

The horizontal maps are the linear isomorphisms obtained by symmetrization. Since the natural map from $S(L_+)$ to $S(L)$ is injective, so is the right-hand vertical map, and its dual is surjective. ■

A *Poisson homogeneous space* (X, P_X) for a Poisson-Lie group (Γ, P_Γ) is a Poisson formal manifold (X, P_X), equipped with a transitive action of Γ, and such that the map $\Gamma \times X \to X$ is Poisson. Then there is a Lie subgroup Γ' of Γ such that $X = \Gamma/\Gamma'$. Following [8], we observe that these two conditions are equivalent: a) the projection map $\Gamma \to X$ is Poisson, b) the Poisson bivector P_X vanishes at one point of X. If either of these equivalent conditions is satisfied, (X, P_X) is said to be *of group type*. (See [11].)

1.4. Poisson homogeneous space structure on G/H

Let $(\mathfrak{g}, r, \langle\ ,\ \rangle_\mathfrak{g}, \mathfrak{h})$ be a quasitriangular Manin pair. Let H be the formal subgroup of G with Lie algebra $\mathfrak{h}$, and let $P_{G/H}$ be the 2-tensor on G/H equal to the projection of r^L.

Proposition 1.1 *Let $(\mathfrak{g}, r, \langle\ ,\ \rangle_\mathfrak{g}, \mathfrak{h})$ be a quasitriangular Manin pair. Then $P_{G/H}$ is a Poisson bivector on G/H, and G/H is a Poisson homogeneous space for (G, P_G).*

Proof. The only nonobvious property is the antisymmetry of the bracket defined by r^L. Let t denote the symmetric element of $\mathfrak{g} \otimes \mathfrak{g}$ defined by $\langle\ ,\ \rangle_\mathfrak{g}$. If f_1 and f_2 are right $\mathfrak{h}$-invariant functions on G, $\{f_1, f_2\} + \{f_2, f_1\} = t^L(df_1 \otimes df_2) = t^R(df_1 \otimes df_2)$, by the invariance of t. Since $\mathfrak{h}$ is Lagrangian, t belongs to $\mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h}$. Since moreover f_1 and f_2 are $\mathfrak{h}$-invariant, $\{f_1, f_2\} + \{f_2, f_1\}$ vanishes. ■

Remark 1 In the case where $(\mathfrak{g}, r, \langle\ ,\ \rangle_\mathfrak{g}, \mathfrak{h})$ corresponds to a Manin quadruple, this statement has been proved by Etingof and Kazhdan, who also constructed a quantization of this Poisson homogeneous space ([11]).

Remark 2 For g in G and x in $\mathfrak{g}$, let us denote the adjoint action of g on x by ${}^g x$. In the case of a quadruple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$, the Poisson homogeneous space $(G/H, P_{G/H})$ is of group type if and only if there exists g in G such that ${}^g\mathfrak{h}$ is *graded* for the Manin triple decomposition, i.e., such that ${}^g\mathfrak{h} = ({}^g\mathfrak{h} \cap \mathfrak{g}_+) \oplus ({}^g\mathfrak{h} \cap \mathfrak{g}_-)$.

1.5. Poisson homogeneous space structure on $G_+/(G_+\cap H)$

Let $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$ be a Manin quadruple. The inclusion $G_+\subset G$ induces an inclusion map $i: G_+/(G_+\cap H)\to G/H$. On the other hand, by Proposition 1.1, $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$ defines a Poisson structure on G/H.

Proposition 1.2 *There exists a unique Poisson structure $P_{G_+/(G_+\cap H)}$ on $G_+/(G_+\cap H)$ such that the inclusion i is Poisson. Then $(G_+/(G_+\cap H),P_{G_+/(G_+\cap H)})$ is a Poisson homogeneous space for (G_+,P_{G_+}).*

Proof. Let $\langle\ ,\ \rangle_{\mathfrak{g}\otimes\mathfrak{g}}$ denote the bilinear form on $\mathfrak{g}\otimes\mathfrak{g}$ defined as the tensor square of $\langle\ ,\ \rangle_{\mathfrak{g}}$. We must prove that for any g in G_+, $r_{\mathfrak{g}_+,\mathfrak{g}_-}$ belongs to $\mathfrak{g}_+\otimes\mathfrak{g}_+ +\mathfrak{g}\otimes{}^g\mathfrak{h}+{}^g\mathfrak{h}\otimes\mathfrak{g}$. The annihilator of this space in $\mathfrak{g}\otimes\mathfrak{g}$ with respect to $\langle\ ,\ \rangle_{\mathfrak{g}\otimes\mathfrak{g}}$ is $(\mathfrak{g}_+\cap{}^g\mathfrak{h})\otimes{}^g\mathfrak{h}+{}^g\mathfrak{h}\otimes(\mathfrak{g}_+\cap{}^g\mathfrak{h})$. Let us show that for any x in this annihilator, $\langle r_{\mathfrak{g}_+,\mathfrak{g}_-},x\rangle_{\mathfrak{g}\otimes\mathfrak{g}}$ is zero. Assume that $x=v\otimes w$, where $v\in{}^g\mathfrak{h}, w\in\mathfrak{g}_+\cap{}^g\mathfrak{h}$; then

$$\langle r_{\mathfrak{g}_+,\mathfrak{g}_-},x\rangle_{\mathfrak{g}\otimes\mathfrak{g}}=\langle\sum_i e^i\otimes e_i,v\otimes w\rangle_{\mathfrak{g}\otimes\mathfrak{g}}=\langle v,w\rangle_{\mathfrak{g}}=0,$$

where the second equality follows from the facts that $\mathfrak{g}_+$ is isotropic and that $(e^i),(e_i)$ are dual bases, and the last equality follows from the isotropy of ${}^g\mathfrak{h}$. On the other hand, the isotropy of $\mathfrak{g}_+$ implies that $\langle r_{\mathfrak{g}_+,\mathfrak{g}_-},(\mathfrak{g}_+\cap{}^g\mathfrak{h})\otimes{}^g\mathfrak{h}\rangle_{\mathfrak{g}\otimes\mathfrak{g}}=0$. Therefore $r_{\mathfrak{g}_+,\mathfrak{g}_-}$ belongs to $\mathfrak{g}_+\otimes\mathfrak{g}_+ +\mathfrak{g}\otimes{}^g\mathfrak{h}+{}^g\mathfrak{h}\otimes\mathfrak{g}$; this implies the first part of the Proposition.

Let us prove that $(G_+/(G_+\cap H),P_{G_+/(G_+\cap H)})$ is a Poisson homogeneous space for (G_+,P_{G_+}). We have a commutative diagram

$$\begin{tikzcd}
G_+\times(G_+/(G_+\cap H)) \arrow[r, "a_{G_+}"] \arrow[d, "i_G\times i"'] & G_+/(G_+\cap H) \arrow[d, "i"'] \\
G\times(G/H) \arrow[r, "a_G"] & G/H
\end{tikzcd}$$

where i_G is the inclusion map of G_+ in G and a_G (resp., a_{G_+}) is the action map of G on G/H (resp., of G_+ on $G_+/(G_+\cap H)$). The maps $i_G\times i$, i and a_G are Poisson maps; since i induces an injection of tangent spaces, it follows that a_{G_+} is a Poisson map. ∎

In [8], Drinfeld defined a Poisson bivector $P'_{G_+/(G_+\cap H)}$ on $G_+/(G_+\cap H)$, which can be described as follows. When V is a Lagrangian subspace of $\mathfrak{g}$, identify $(\mathfrak{g}_+\cap V)^\perp$ with a subspace of $\mathfrak{g}_-$, and define $\bar{\xi}_V:(\mathfrak{g}_+\cap V)^\perp\to\mathfrak{g}_+/(\mathfrak{g}_+\cap V)$ to be the linear map which, to any element a_- of $(\mathfrak{g}_+\cap V)^\perp\subset\mathfrak{g}_-$, associates the class of an element $a_+\in\mathfrak{g}_+$ such that $a_+ + a_-$ belongs to V. Then there is a unique element $\xi_V\in(\mathfrak{g}_+/(\mathfrak{g}_+\cap V))^{\otimes 2}$, such that $(a_-\otimes id)(\xi_V)=\bar{\xi}_V(a_-)$, for any $a_-\in(\mathfrak{g}_+\cap V)^\perp$.

For any g in G_+, identify the tangent space of $G_+/(G_+\cap H)$ at $g(G_+\cap H)$ with $\mathfrak{g}_+/(\mathfrak{g}_+\cap{}^g\mathfrak{h})$ via left-invariant vector fields. The element ξ_g of $(\mathfrak{g}_+/(\mathfrak{g}_+\cap{}^g\mathfrak{h}))^{\otimes 2}$ corresponding to the value of $P'_{G_+/(G_+\cap H)}$ at $g(G_+\cap H)$ is then $\xi_{{}^g\mathfrak{h}}$.

Proposition 1.3 *The bivectors* $P_{G_+/(G_+\cap H)}$ *and* $P'_{G_+/(G_+\cap H)}$ *are equal.*

Proof. We have to show that the injection $i : G_+/(G_+\cap H) \to G/H$ is compatible with the bivectors $P'_{G_+/(G_+\cap H)}$ and $P_{G/H}$. The differential of i at $g(G_+\cap H)$, where g belongs to G_+, induces the canonical injection ι from $\mathfrak{g}_+/(\mathfrak{g}_+\cap{}^g\mathfrak{h})$ to $\mathfrak{g}/{}^g\mathfrak{h}$. Let us show that for any g in G_+, the injection $\iota\otimes\iota$ maps $\xi_g\in(\mathfrak{g}_+/(\mathfrak{g}_+\cap{}^g\mathfrak{h}))^{\otimes 2}$ to the class of $r_{\mathfrak{g}_+,\mathfrak{g}_-}$ in $(\mathfrak{g}/{}^g\mathfrak{h})^{\otimes 2}$. We have to verify the commutativity of the diagram

$$\begin{tikzcd}
(\mathfrak{g}_+\cap{}^g\mathfrak{h})^\perp \arrow[d, "\bar{\xi}_g"] & ({}^g\mathfrak{h})^\perp \arrow[l, "\iota^*"'] \arrow[d, "\bar{r}_g"] \\
\mathfrak{g}_+/(\mathfrak{g}_+\cap{}^g\mathfrak{h}) \arrow[r, "\iota"] & \mathfrak{g}/{}^g\mathfrak{h}
\end{tikzcd}$$

where the horizontal maps are the natural injection and restriction maps, and $\bar{r}_g$ is defined by $\bar{r}_g(a) =$ the class of $\langle r_{\mathfrak{g}_+,\mathfrak{g}_-}, a\otimes id\rangle_{\mathfrak{g}\otimes\mathfrak{g}}$ mod ${}^g\mathfrak{h}$, for $a\in({}^g\mathfrak{h})^\perp$. Let a belong to $({}^g\mathfrak{h})^\perp$. By the maximal isotropy of ${}^g\mathfrak{h}$, the element a can be identified with an element of ${}^g\mathfrak{h}$. Let us write $a = a_+ + a_-$, with $a_\pm\in\mathfrak{g}_\pm$. Then $\iota^*(a) = a_-$, $\bar{\xi}_g(a_-) = a_+ + (\mathfrak{g}_+\cap{}^g\mathfrak{h})$ by the definition of $\bar{\xi}_g$ and because $a\in{}^g\mathfrak{h}$, and $\iota(a_+ + \mathfrak{g}_+\cap{}^g\mathfrak{h}) = a_+ + {}^g\mathfrak{h}$. On the other hand, $\bar{r}_g(a) = a_+ + {}^g\mathfrak{h}$, so the diagram commutes. ■

Moreover, it is a result of Drinfeld ([8], Remark 2) that the formal Poisson homogeneous spaces for (G_+, P_{G_+}) are all of the type described in Proposition 1.2.

Remark 3 The (G_+, P_{G_+})-Poisson homogeneous space

$$(G_+/(G_+\cap H), P_{G_+/(G_+\cap H)})$$

is of group type if and only if for some $g\in G_+$, ${}^g\mathfrak{h}$ is graded for the Manin triple decomposition, see Remark 2.

Remark 4 *Conjugates of Manin quadruples.* Let us denote the conjugate of an element x in G by an element h in G by ${}^gx = gxg^{-1}$. For $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ a Manin quadruple, and g in G, $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, {}^g\mathfrak{h})$ is also a Manin quadruple. If g belongs to H, this is the same quadruple; and if g belongs to G_+, the Poisson structure induced by $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, {}^g\mathfrak{h})$ on $G_+/(G_+\cap{}^gH)$ is isomorphic to that induced by $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ on $G_+/(G_+\cap H)$, via conjugation by g. It follows that Poisson homogeneous space structures induced by $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ are indexed by elements of the double quotient $G_+\backslash G/H$.

Remark 5 The first examples of Poisson homogeneous spaces which are not of group type were studied in [3] and [13], under the name of affine Poisson structures. In these cases, the stabilizer of a point is trivial.

2. Quantization of G/H

In this section, we introduce axioms for the quantization of a quasitriangular Manin pair. We then show that any solution to this quantization problem leads to a quantization of the Poisson homogeneous space G/H constructed in Section 1.4.

2.1. Definition of quantization of Poisson homogeneous spaces

Let (Γ, P_Γ) be a Poisson-Lie group and let (X, P_X) be a formal Poisson homogeneous space over (Γ, P_Γ). Let $(\mathscr{A}, \Delta_{\mathscr{A}})$ be a quantization of the enveloping algebra of the Lie algebra of Γ, and let $\mathscr{A}^{opp}$ denote the opposite algebra to $\mathscr{A}$.

Definition 2.1 *A quantization of the Poisson homogeneous space (X, P_X) is a $\mathbb{C}[[\hbar]]$-algebra $\mathscr{X}$, such that*
1) $\mathscr{X}$ is a quantization of the Poisson algebra $(\mathscr{O}_X, P_X)$ of formal functions on X, and
2) $\mathscr{X}$ is equipped with an algebra-module structure over $(\mathscr{A}^{opp}, \Delta)$, whose reduction mod $\hbar$ coincides with the algebra-module structure of $\mathscr{O}_X$ over $((U\mathfrak{g})^{opp}, \Delta_0)$, where Δ_0 is the coproduct of $U\mathfrak{g}$.

Condition 1 means that there is an isomorphism of $\mathbb{C}[[\hbar]]$-modules from $\mathscr{X}$ to $\mathscr{O}_X[[\hbar]]$, inducing an algebra isomorphism between $\mathscr{X}/\hbar\mathscr{X}$ and $\mathscr{O}_X$, and inducing on $\mathscr{O}_X$ the Poisson structure defined by P_X.

The first part of condition 2 means that $\mathscr{X}$ has a module structure over $\mathscr{A}^{opp}$, such that for $x, y \in \mathscr{X}, a \in \mathscr{A}$, and $\Delta(a) = \sum a^{(1)} \otimes a^{(2)}$, $a(xy) = \sum a^{(1)}(x) a^{(2)}(y)$.

Conventions

We will say that a $\mathbb{C}[[\hbar]]$-module V is *topologically free* if it is isomorphic to $W[[\hbar]]$, where W is a complex vector space. We denote the canonical projection of V onto $V/\hbar V$ by $v \mapsto v$ mod $\hbar$. In what follows, all tensor products of $\mathbb{C}[[\hbar]]$-modules are $\hbar$-adically completed. When E is a $\mathbb{C}[[\hbar]]$-module, we denote by E^* its dual $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(E, \mathbb{C}[[\hbar]])$. For a subset $\mathscr{S}$ of an algebra $\mathscr{A}$, we denote by $\mathscr{S}^\times$ the group of invertible elements of $\mathscr{S}$. When $\mathscr{A}, \mathscr{B}$ are two Hopf or quasi-Hopf algebras with unit elements $1_{\mathscr{A}}, 1_{\mathscr{B}}$ and counit maps $\varepsilon_{\mathscr{A}}, \varepsilon_{\mathscr{B}}$, and $\mathscr{S}$ is a subset of $\mathscr{A} \otimes \mathscr{B}$, we denote by $\mathscr{S}_0^\times$ the subgroup of $\mathscr{S}^\times$ with elements x such that $(\varepsilon_{\mathscr{A}} \otimes id)(x) = 1_{\mathscr{B}}$ and $(id \otimes \varepsilon_{\mathscr{B}})(x) = 1_{\mathscr{A}}$. We also denote $\mathrm{Ker}\,\varepsilon_{\mathscr{A}}$ by $\mathscr{A}_0$.

2.2. Quantization of quasitriangular Manin pairs

Definition 2.2 *A quantization of a quasitriangular Manin pair $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$ is the data of*
1) a quasitriangular Hopf algebra $(A, \Delta, \mathscr{R})$ quantizing $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}})$,
2) a subalgebra $B \subset A$ and an element F in $(A \otimes A)_0^\times$, such that
  a) $B \subset A$ is a flat deformation of the inclusion $U\mathfrak{h} \subset U\mathfrak{g}$,
  b) $F\Delta(B)F^{-1} \subset B \otimes B$, and $F^{(12)}(\Delta \otimes id)(F)\left(F^{(23)}(id \otimes \Delta)(F)\right)^{-1} \in B^{\otimes 3}$,
  c) there exists a Lagrangian complement L of $\mathfrak{h}$ in $\mathfrak{g}$, such that

$$\left(\frac{1}{\hbar}(F - F^{(21)}) \text{ mod } \hbar\right) = r - r_{\mathfrak{h},L},$$

where $r_{\mathfrak{h},L} = \sum_i \varepsilon^i \otimes \varepsilon_i$, *and* $(\varepsilon^i), (\varepsilon_i)$ *are dual bases of* $\mathfrak{h}$ *and* L.

Observe that condition 1 implies that $(\frac{1}{\hbar}(\mathscr{R} - 1) \bmod \hbar) = r$. In condition 2, $(A \otimes A)_0^\times = 1 + \hbar(A_0 \otimes A_0)$. Condition 2a means that the $\mathbb{C}[[\hbar]]$-module isomorphism between A and $U\mathfrak{g}[[\hbar]]$ arising from condition 1 can be chosen in such a way that it induces an isomorphism between B and $U\mathfrak{h}[[\hbar]]$.

Let (A, Δ_B, Φ_B) be the quasi-Hopf algebra obtained by twisting the Hopf algebra (A, Δ) by F. Then, by definition, $\Delta_B(x) = F\Delta(x)F^{-1}$, for any $x \in A$, and $\Phi_B = F^{(12)}(\Delta \otimes id)(F)\left(F^{(23)}(id \otimes \Delta)(F)\right)^{-1}$. Condition 2b expresses the fact that $B \subset A$ is a sub-quasi-Hopf algebra of A.

Condition 2c expresses the fact that the classical limit of (B, Δ_B, Φ_B) is $\mathfrak{h}$ equipped with the Lie quasi-bialgebra structure associated with L.

2.3. Quantization of $(G/H, P_{G/H})$

Let us denote the counit map of A by ε.

Theorem 2.1 *Assume that* $((A, \Delta, \mathscr{R}), B, F)$ *is a quantization of the quasitriangular Manin pair* $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$. *Let* $(A^*)^B$ *be the subspace of* A^* *consisting of the forms* ℓ *on* A *such that* $\ell(ab) = \ell(a)\varepsilon(b)$ *for any* $a \in A$ *and* $b \in B$.

a) For ℓ, ℓ' *in* $(A^*)^B$, *define* $\ell * \ell'$ *to be the element of* A^* *such that*

$$(\ell * \ell')(a) = (\ell \otimes \ell')(\Delta(a)F^{-1}),$$

for any a *in* A. *Then* $*$ *defines an associative algebra structure on* $(A^*)^B$.

b) For $a \in A$ *and* $\ell \in (A^*)^B$, *define* $a\ell$ *to be the form on* A *such that* $(a\ell)(a') = \ell(aa')$, *for any* $a' \in A$. *This map defines on* $((A^*)^B, *)$ *a structure of an algebra-module over the Hopf algebra* (A^{opp}, Δ).

c) The algebra $((A^*)^B, *)$ *is a quantization of the Poisson algebra* $(\mathscr{O}_{G/H}, P_{G/H})$.

With its algebra-module structure over (A^{opp}, Δ), $((A^*)^B, *)$ *is a quantization of the Poisson homogeneous space* $(G/H, P_{G/H})$.

Proof. Let ℓ and ℓ' belong to $(A^*)^B$. Then for any $a \in A, b \in B$,

$$\begin{aligned}(\ell * \ell')(ab) &= (\ell \otimes \ell')(\Delta(a)\Delta(b)F^{-1}) = (\ell \otimes \ell')(\Delta(a)F^{-1}\Delta_B(b)) \\ &= \varepsilon(b)(\ell \otimes \ell')(\Delta(a)F^{-1}) = \varepsilon(b)(\ell * \ell')(a),\end{aligned}$$

where the third equality follows from the fact that $\Delta_B(B) \subset B \otimes B$ and $(\varepsilon \otimes \varepsilon) \circ \Delta_B = \varepsilon$. It follows that $\ell * \ell'$ belongs to $(A^*)^B$.

Let ℓ, ℓ' and ℓ'' belong to $(A^*)^B$. Then for any a in A,

$$((\ell * \ell') * \ell'')(a) = (\ell \otimes \ell' \otimes \ell'')\left((\Delta \otimes id) \circ \Delta(a)(\Delta \otimes id)(F^{-1})(F^{(12)})^{-1}\right), \quad (2)$$

and

$$(\ell * (\ell' * \ell''))(a) = (\ell \otimes \ell' \otimes \ell'')\left((id \otimes \Delta) \circ \Delta(a)(id \otimes \Delta)(F^{-1})(F^{(23)})^{-1}\right).$$

By the coassociativity of Δ and the definition of Φ_B, this expression is equal to $(\ell\otimes\ell'\otimes\ell'')\big((\Delta\otimes id)\circ\Delta(a)(\Delta\otimes id)(F^{-1})(F^{(12)})^{-1}\Phi_B^{-1}\big)$, and since Φ_B belongs to $B^{\otimes 3}$ and $\varepsilon^{\otimes 3}(\Phi_B)=1$, this is equal to the right-hand side of (2). It follows that $(\ell*\ell')*\ell''=\ell*(\ell'*\ell'')$, so $*$ is associative. Moreover, ε belongs to $(A^*)^B$ and is the unit element of $((A^*)^B,*)$. This proves part a of the theorem.

Part b follows from the definitions.

Let us prove that $(A^*)^B$ is a flat deformation of $(U\mathfrak{g}/(U\mathfrak{g})\mathfrak{h})^*$. Let us consider a Lagrangian complement L of $\mathfrak{h}$ in $\mathfrak{g}$. Let Sym denote the symmetrisation map from the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ to $U\mathfrak{g}$, i.e., the unique linear map such that $\mathrm{Sym}(x^l)=x^l$ for any $x\in\mathfrak{g}$ and $l\geq 0$, and let us define $\widetilde{L}$ to be $\mathrm{Sym}(S(L))$. Thus $\widetilde{L}$ is a linear subspace of $U\mathfrak{g}$, and inclusion of $\widetilde{L}\otimes U\mathfrak{h}$ in $U\mathfrak{g}\otimes U\mathfrak{g}$ followed by multiplication induces a linear isomorphism from $\widetilde{L}\otimes U\mathfrak{h}$ to $U\mathfrak{g}$. It follows that the restriction to $\widetilde{L}$ of the projection $U\mathfrak{g}\to U\mathfrak{g}/(U\mathfrak{g})\mathfrak{h}$ is an isomorphism, which defines a linear isomorphism between $\mathscr{O}_{G/H}=(U\mathfrak{g}/(U\mathfrak{g})\mathfrak{h})^*$ and $\widetilde{L}^*$.

Let us fix an isomorphism of $\mathbb{C}[[\hbar]]$-modules from A to $U\mathfrak{g}[[\hbar]]$, inducing an isomorphism between B and $U\mathfrak{h}[[\hbar]]$ and let us define C to be the preimage of $\widetilde{L}[[\hbar]]$. Thus C is isomorphic to $\widetilde{L}[[\hbar]]$, and inclusion followed by multiplication induces a linear isomorphism between $C\otimes B$ and A, as does any morphism between two topologically free $\mathbb{C}[[\hbar]]$-modules E and F, which induces an isomorphism between $E/\hbar E$ and $F/\hbar F$. Therefore, restriction of linear forms to C induces an isomorphism between $(A^*)^B$ and C^*. It follows that $(A^*)^B$ is isomorphic to $(\widetilde{L}[[\hbar]])^*$, which is in turn isomorphic to $\widetilde{L}^*[[\hbar]]$. Since $\widetilde{L}^*$ is isomorphic to $\mathscr{O}_{G/H}$, $(A^*)^B$ is isomorphic to $\mathscr{O}_{G/H}[[\hbar]]$.

Let us fix ℓ,ℓ' in $(A^*)^B$, and let us compute $(\frac{1}{\hbar}(\ell*\ell'-\ell'*\ell) \bmod \hbar)$. Let us set $f=(\frac{1}{\hbar}(F-1) \bmod \hbar)$. For a in A,

$$\begin{aligned}\frac{1}{\hbar}(\ell*\ell'-\ell'*\ell)(a)&=(\ell\otimes\ell')\left(\frac{1}{\hbar}(\Delta(a)F^{-1}-\Delta'(a)(F^{(21)})^{-1})\right)\\&=(\ell\otimes\ell')\left(\frac{1}{\hbar}(\Delta(a)F^{-1}-\mathscr{R}\Delta(a)\mathscr{R}^{-1}(F^{(21)})^{-1})\right)\\&=(\ell\otimes\ell')\left(-r\Delta_0(a_0)+\Delta_0(a_0)(r+f^{(21)}-f)\right)+o(\hbar),\end{aligned}$$

where Δ_0 is the coproduct of $U\mathfrak{g}$ and a_0 is the image of a in $A/\hbar A=U\mathfrak{g}$. Since, by condition 2c of Definition 2.2, $r-f+f^{(21)}$ is equal to $r_{\mathfrak{h},L}$, it belongs to $\mathfrak{h}\otimes\mathfrak{g}$ and since ℓ and ℓ' are right B-invariant, $(\frac{1}{\hbar}(\ell*\ell'-\ell'*\ell)(a) \bmod \hbar)=(\ell\otimes\ell')(-r\Delta_0(a_0))$, which is the Poisson bracket defined by r^L on G/H. This ends the proof of part c of the theorem.

Since $(F \bmod \hbar)=1$, the reduction modulo $\hbar$ of the algebra-module structure of $((A^*)^B,*)$ over (A^{opp},Δ) is that of $\mathscr{O}_X$ over $((U\mathfrak{g})^{opp},\Delta_0)$. This ends the proof of the theorem. ■

Remark 6 If $((A,\Delta,\mathscr{R}),B,F)$ is a quantization of a quasitriangular Manin pair $(\mathfrak{g},r,\langle\ ,\ \rangle_{\mathfrak{g}},\mathfrak{h})$, and if F_0 is an element of $(B\otimes B)_0^\times$, then $((A,\Delta,\mathscr{R}),F_0F)$ is a quantization of the same Manin pair. We observe that the product $*$ on $(A^*)^B$ is independent of such a modification of F.

Remark 7 In the case of a Manin quadruple $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$ where $\mathfrak{h}$ is graded for the Manin triple decomposition (see Remark 2), $r_{\mathfrak{g}_+,\mathfrak{g}_-}-r_{\mathfrak{h},L}=f-f^{(21)}$ belongs to $\mathfrak{h}\otimes\mathfrak{g}+\mathfrak{g}\otimes\mathfrak{h}$. The corresponding quantum condition is that

$$F\in 1+\hbar(B_0\otimes A_0+A_0\otimes B_0). \tag{3}$$

This is the case is [9], where F belongs to $(B\otimes A)_0^\times=1+\hbar(B_0\otimes A_0)$.

When condition (3) is fulfilled, the product $*$ is the restriction to $(A^*)^B$ of the usual product on A^*, defined as the dual map to Δ.

2.4. Relations in $(A^*)^B$

It is well known that the matrix coefficients of the representations of a quasitriangular Hopf algebra can be organized in L-operators, satisfying the so-called *RLL* relations. We recall this construction and introduce analogues of these matrix coefficients and of the *RLL* relations for the algebra $(A^*)^B$.

Recall that there is an algebra structure on A^*, where the product is $(\ell,\ell')\mapsto\ell\ell'$, such that for any $a\in A$, $(\ell\ell')(a)=(\ell\otimes\ell')(\Delta(a))$.

Let $\mathrm{Rep}(A)$ be the category of modules over A, which are free and finite dimensional over $\mathbb{C}[[\hbar]]$. There is a unique map

$$\oplus_{V\in\mathrm{Rep}(A)}(V^*\otimes V)\to A^*,\quad \kappa\mapsto\ell_\kappa,$$

such that for any object (V,π_V) in $\mathrm{Rep}(A)$, and any $\xi\in V^*$ and $v\in V$, $\ell_{\xi\otimes v}(a)=\xi(\pi_V(a)v)$, for any $a\in A$. Define $\mathrm{Coeff}(A)$ to be the image of this map. Then $\mathrm{Coeff}(A)$ is a subalgebra of A^*. Moreover, there is a Hopf algebra structure on $\mathrm{Coeff}(A)$, with coproduct $\Delta_{\mathrm{Coeff}(A)}$ and counit $\varepsilon_{\mathrm{Coeff}(A)}$, uniquely determined by the rules

$$\Delta_{\mathrm{Coeff}(A)}(\ell_{\xi\otimes v})=\sum_i\ell_{\xi\otimes v_i}\otimes\ell_{\xi^i\otimes v},\quad \varepsilon_{\mathrm{Coeff}(A)}(\ell_{\xi\otimes v})=\xi(v),$$

where (v_i) and (ξ^i) are dual bases of V and V^*. The duality pairing between A and A^* then induces a Hopf algebra pairing between (A,Δ) and $(\mathrm{Coeff}(A),\Delta_{\mathrm{Coeff}(A)})$ (see [1]).

For V an object of $\mathrm{Rep}(A)$, define L_V to be the element of $\mathrm{End}(V)\otimes\mathrm{Coeff}(A)$ equal to $\sum_i\kappa^i\otimes\ell_{\kappa_i}$, where (κ^i) and (κ_i) are dual bases of $\mathrm{End}(V)$ and $V^*\otimes V$. It follows from $\mathscr{R}\Delta=\Delta'\mathscr{R}$ that the relation

$$R_{V,W}^{(12)}L_V^{(1a)}L_W^{(2a)}=L_W^{(2a)}L_V^{(1a)}R_{V,W}^{(12)}$$

is satisfied in $\mathrm{End}(V)\otimes\mathrm{End}(W)\otimes\mathrm{Coeff}(A)$, where the superscripts 1, 2, and a refer to the successive factors of the tensor product. Moreover,

$$(id_V \otimes \Delta_{\mathrm{Coeff}(A)})(L_V) = L_V^{(1a)} L_V^{(1a')}$$

holds in $\mathrm{End}(V)\otimes\mathrm{Coeff}(A)^{\otimes 2}$, where the superscripts $1, a$ and a' refer to the successive factors of this tensor product.

For (V,π_V) an A-module, we set $V^B = \{v\in V | \forall b\in B, \pi_V(b)(v) = \varepsilon(b)v\}$. There is a unique map

$$\oplus_{V\in\mathrm{Rep}(A)}(V^*\otimes V^B)\to (A^*)^B, \quad \kappa\mapsto\widetilde{\ell}_\kappa$$

such that for $\xi\in V^*$ and $v\in V^B$, $\widetilde{\ell}_{\xi\otimes v}(a) = \xi(\pi_V(a)v)$, for any $a\in A$. Define $\mathrm{Coeff}(A,B)$ to be the image of this map.

For V an object of $\mathrm{Rep}(A)$, V^B is a free, finite-dimensional $\mathbb{C}[[\hbar]]$-module. It follows that the dual of $V^*\otimes V^B$ is $(V^B)^*\otimes V$, which may be identified with $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(V^B,V)$. Define $\widetilde{L}_V$ to be the element of $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(V^B,V)\otimes\mathrm{Coeff}(A,B)$ equal to $\sum_i \kappa^i\otimes\widetilde{\ell}_{\kappa_i}$, where (κ^i) and (κ_i) are dual bases of $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(V^B,V)$ and $V^*\otimes V^B$.

When (V,π_V) and (W,π_W) are objects of $\mathrm{Rep}(A)$, let $R_{V,W}$ be the element of $\mathrm{End}_{\mathbb{C}[[\hbar]]}(V\otimes W)$ equal to $(\pi_V\otimes\pi_W)(\mathscr{R})$, where $\mathscr{R}$ is the R-matrix of A. Recall that the twist of $\mathscr{R}$ by F is $\mathscr{R}_B = F^{(21)}\mathscr{R}F^{-1}$, and set $R_{B;V,W} = (\pi_V\otimes\pi_W)(\mathscr{R}_B)$.

Proposition 2.1 $\mathrm{Coeff}(A,B)$ *is a subalgebra of* $(A^*)^B$. *For any objects V and W in* $\mathrm{Rep}(A)$, *the relation*

$$R^{(12)}_{V,W}\widetilde{L}^{(1a)}_V\widetilde{L}^{(2a)}_W = \widetilde{L}^{(2a)}_W\widetilde{L}^{(1a)}_V(R^{(12)}_{B;V,W})_{|Z}$$

is satisfied in $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(Z,V\otimes W)\otimes\mathrm{Coeff}(A,B)$, *where Z is the intersection* $(V^B\otimes W^B)\cap R^{-1}_{B;V,W}(V^B\otimes W^B)$. *In this equality, the left-hand side is an element of* $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(V^B\otimes W^B, V\otimes W)\otimes\mathrm{Coeff}(A,B)$, *viewed as an element of* $\mathrm{Hom}_{\mathbb{C}[[\hbar]]}(Z,V\otimes W)\otimes\mathrm{Coeff}(A,B)$ *by restriction.*

Recall that an algebra-comodule $\mathscr{X}$ over a Hopf algebra $(\mathscr{A},\Delta_{\mathscr{A}})$ is the data of an algebra structure over $\mathscr{X}$ and a left comodule structure of $\mathscr{X}$ over $(\mathscr{A},\Delta_{\mathscr{A}})$, $\Delta_{\mathscr{X},\mathscr{A}}:\mathscr{X}\to\mathscr{A}\otimes\mathscr{X}$, which is also a morphism of algebras.

Proposition 2.2 *There is a unique algebra-comodule structure on* $\mathrm{Coeff}(A,B)$ *over* $(\mathrm{Coeff}(A),\Delta_{\mathrm{Coeff}(A)})$, *compatible with the algebra-module structure of* $(A^*)^B$ *over* (A^{opp},Δ). *The relation*

$$(id_V\otimes\Delta_{\mathrm{Coeff}(A,B),\mathrm{Coeff}(A)})(\widetilde{L}_V) = L_V^{(1a)}\widetilde{L}_V^{(1a')}$$

is satisfied in $\mathrm{End}(V)\otimes\mathrm{Coeff}(A)\otimes\mathrm{Coeff}(A,B)$, *where the superscripts* 1, *a, and a′ refer to the successive factors of this tensor product.*

3. Quantization of $G_+/(G_+ \cap H)$

In this section, we state axioms for the quantization of a Manin quadruple, and show that any such quantization gives rise to a quantization of the Poisson homogeneous space $G_+/(G_+ \cap H)$ constructed in [8] (see Propositions 1.2 and 1.3).

3.1. Quantization of Manin quadruples

Let us fix a Manin quadruple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$. Recall that $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ gives rise to a quasitriangular Manin pair $(\mathfrak{g}, r, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h})$, if we set $r = r_{\mathfrak{g}_+,\mathfrak{g}_-} = \sum_i e^i \otimes e_i$, where (e^i) and (e_i) are dual bases of $\mathfrak{g}_+$ and $\mathfrak{g}_-$.

Definition 3.1 *A quantization of a Manin quadruple* $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, \mathfrak{h})$ *is the data of*
1) a quantization $((A, \Delta, \mathscr{R}), B, F)$ *of the quasitriangular Manin pair*

$$(\mathfrak{g}, r_{\mathfrak{g}_+,\mathfrak{g}_-}, \langle\ ,\ \rangle_{\mathfrak{g}}, \mathfrak{h}),$$

2) a Hopf subalgebra A_+ *of* (A, Δ) *such that*
- *a)* $A_+ \subset A$ *is a flat deformation of* $U\mathfrak{g}_+ \subset U\mathfrak{g}$,
- *b)* $B \cap A_+ \subset A_+$ *is a flat deformation of the inclusion* $U(\mathfrak{h} \cap \mathfrak{g}_+) \subset U\mathfrak{g}_+$,
- *c)* F *satisfies*

$$F^{-1} \in \left((AB_0 + A_+) \otimes A + A \otimes AB_0\right) \cap \left(AB_0 \otimes A + A \otimes (AB_0 + A_+)\right). \qquad (4)$$

It follows from condition 2c of Definition 2.2 and the beginning of the proof of Proposition 1.2 that $(\frac{1}{\hbar}(F - F^{(21)}) \bmod \hbar)$ belongs to $\mathfrak{g}_+ \otimes \mathfrak{g}_+ + \mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h} = ((\mathfrak{h} + \mathfrak{g}_+) \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h}) \cap (\mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes (\mathfrak{h} + \mathfrak{g}_+))$. Therefore condition (4) is natural. It is equivalent to the condition that F belong to the product of subgroups of $(A \otimes A)_0^\times$

$$\left(1 + \hbar(AB_0 \otimes A_0 + A_0 \otimes AB_0)\right)\left(1 + \hbar(A_+)_0 \otimes (A_+)_0\right).$$

Example
Recall that (1) is a graded Manin quadruple. In [9], a quantization of this quadruple was constructed for the case where $\bar{\mathfrak{g}} = \mathfrak{sl}_2$.

Remark 8 In the case where $\mathfrak{g}_+ \cap \mathfrak{h} = 0$, which corresponds to a homogeneous space over G_+ with trivial stabilizer, F automatically satisfies condition (4). Indeed, in that case, multiplication induces an isomorphism $A_+ \otimes B \to A$, therefore $A = AB_0 + A_+$.

3.2. Quantization of $(G_+/(G_+ \cap H), P_{G_+/(G_+ \cap H)})$

Let I_0 be the subspace of $\mathscr{O}_{G/H}$ equal to $\mathscr{O}_{G/H} \cap (U\mathfrak{g}_+)^\perp$. It follows from Lemma 1.1 that I_0 is an ideal of $\mathscr{O}_{G/H}$, and that the algebra $\mathscr{O}_{G_+/(G_+ \cap H)}$ can be identified as a Poisson algebra with the quotient $\mathscr{O}_{G/H}/I_0$.

Let I be the subspace of $(A^*)^B$ defined as the set of all linear forms ℓ on A such that $\ell(a_+) = 0$ for any $a_+ \in A_+$. Therefore

$$I = (A^*)^B \cap A_+^\perp.$$

Theorem 3.1 *Assume that* $((A,\Delta,\mathscr{R}),A_+,B,F)$ *is a quantization of the Manin quadruple* $(\mathfrak{g},\mathfrak{g}_+,\mathfrak{g}_-,\mathfrak{h})$*. Then* I *is a two-sided ideal in* $(A^*)^B$*, the algebra* $(A^*)^B/I$ *is a flat deformation of* $\mathscr{O}_{G_+/(G_+\cap H)}$*, and is a quantization of the formal Poisson space* $(G_+/(G_+\cap H),P_{G_+/(G_+\cap H)})$*.*

Moreover, I *is preserved by the action of* A_+^{opp}*, and* $(A^*)^B/I$ *is an algebra-module over* (A_+^{opp},Δ)*. With this algebra-module structure,* $(A^*)^B/I$ *is a quantization of the* (G_+,P_{G_+})*-Poisson homogeneous space* $(G_+/(G_+\cap H),P_{G_+/(G_+\cap H)})$*.*

Proof. Let us fix ℓ in I and ℓ' in $(A^*)^B$. For any a_+ in A_+, we have

$$(\ell*\ell')(a_+)=(\ell\otimes\ell')(\Delta(a_+)F^{-1})=0$$

because $\Delta(A_+)\subset A_+\otimes A_+$ and by assumption (4) on F. In the same way, $(\ell'*\ell)(a_+)=0$. Therefore I is a two-sided ideal in $(A^*)^B$.

Let us prove that $(A^*)^B/I$ is a flat deformation of $\mathscr{O}_{G_+/(G_+\cap H)}$. To this end, we will identify the $\mathbb{C}[[\hbar]]$-modules $(A^*)^B/I$ with $(A_+^*)^{A_+\cap B}$, to which we apply the result of Theorem 2.1.

Recall that B_0 and $(A_+\cap B)_0$ denote the augmentation ideals of B and $A_+\cap B$. Thus $(A^*)^B$ is equal to $(A/AB_0)^*$, where AB_0 is the image of the product map $A\otimes B_0\to A$. In the same way, $(A_+^*)^{A_+\cap B}$ is equal to $(A_+/A_+(A_+\cap B)_0)^*$. Let us show that $(A^*)^B/I$ is equal to $(A_+/A_+(A_+\cap B)_0)^*$. Restriction of a linear form to A_+ induces a linear map $\rho:(A/AB_0)^*\to(A_+/A_+(A_+\cap B)_0)^*$. Moreover, the kernel of ρ is I, therefore ρ induces an injective map

$$\widetilde{\rho}:(A/AB_0)^*/I\to(A_+/A_+(A_+\cap B)_0)^*.$$

Let us now show that $\widetilde{\rho}$ is surjective. For this, it is enough to show that the restriction map $\rho:(A/AB_0)^*\to(A_+/A_+(A_+\cap B)_0)^*$ is surjective. $(A/AB_0)^*$ and $(A_+/A_+(A_+\cap B)_0)^*$ are topologically free $\mathbb{C}[[\hbar]]$-modules, and the map from $(A/AB_0)^*/\hbar(A/AB_0)^*$ to $(A_+/A_+(A_+\cap B)_0)^*/\hbar(A_+/A_+(A_+\cap B)_0)^*$ coincides with the canonical map from $\mathscr{O}_{G/H}$ to $\mathscr{O}_{G_+/(G_+\cap H)}$ which, by Lemma 1.1, is surjective. Therefore ρ is surjective, and so is $\widetilde{\rho}$. It follows that $(A_+^*)^{A_+\cap B}$ is a flat deformation of $\mathscr{O}_{G_+/(G_+\cap H)}$.

There is a commutative diagram of algebras

$$\begin{tikzcd}
(A^*)^B \arrow[r] \arrow[d] & (A^*)^B/I \arrow[d] \\
\mathscr{O}_{G/H} \arrow[r] & \mathscr{O}_{G_+/(G_+\cap H)}
\end{tikzcd}$$

where the vertical maps are projections $X\to X/\hbar X$. Since the projection $\mathscr{O}_{G/H}\to\mathscr{O}_{G_+/(G_+\cap H)}$ is a morphism of Poisson algebras, where $\mathscr{O}_{G/H}$ and $\mathscr{O}_{G_+/(G_+\cap H)}$ are equipped with $P_{G/H}$ and $P_{G_+/(G_+\cap H)}$, the classical limit of $(A^*)^B/I$ is $(\mathscr{O}_{G_+/(G_+\cap H)},P_{G_+/(G_+\cap H)})$.

Finally, the algebra-module structure of $(A^*)^B$ over (A^{opp}, Δ) induces by restriction an algebra-module structure on $(A^*)^B$ over (A_+^{opp}, Δ), and since I is preserved by the action of A_+^{opp}, $(A^*)^B/I$ is also an algebra-module over (A_+^{opp}, Δ). ■

Remark 9 For $u \in A^\times$, set ${}^uB = uBu^{-1}$ and ${}^uF = (u \otimes u)\Delta_B(u)^{-1}F$. Let $((A^*)^{^uB}, *_u)$ be the algebra-module over (A^{opp}, Δ) corresponding to $(A, {}^uB, {}^uF)$. There is an algebra-module isomorphism $i_u : ((A^*)^B, *) \to ((A^*)^{^uB}, *_u)$, given by $(i_u\ell)(a) = \ell(au)$, for any $a \in A$.

If u does not lie in A_+, there is no reason for $A_+ \cap {}^uB$ to be a flat deformation of its classical limit, nor for ${}^uF$ to satisfy (4). But, if the conditions of Theorem 3.1 are still valid for $(A, {}^uB, {}^uF)$, the resulting algebra-module over (A_+^{opp}, Δ_+) can be different from the one arising from (A, B, F). We will see an example of this situation in Section 4.1.

Remark 10 In their study of preferred deformations, Bonneau *et al.* studied the case of quotients of compact, connected Lie groups ([2]). See also the article [15] and references that it cites.

In [5], Donin, Gurevich, and Shnider used quasi-Hopf algebra techniques to construct quantizations of some homogeneous spaces. More precisely, they classified the Poisson homogeneous structures on the semisimple orbits of a simple Lie group with Lie algebra $\mathfrak{g}_0$, and constructed their quantizations using Drinfeld's series $F_\hbar$ relating the Hopf algebra $U_\hbar\mathfrak{g}_0$ to a quasi-Hopf algebra structure on $U\mathfrak{g}_0[[\hbar]]$ involving the Knizhnik-Zamolodchikov associator.

In [16], Parmentier also used twists to propose a quantization scheme of Poisson structures on Lie groups, generalizing the affine Poisson structures.

4. Examples

In view of Remark 7, we can only find nontrivial applications of the above results in the case of a nongraded $\mathfrak{h}$. In this section, we shall construct quantizations of some nongraded Manin quadruples.

4.1. Finite-dimensional examples

Let us set $\bar{\mathfrak{g}} = \mathfrak{sl}_2(\mathbb{C})$; let $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_+ \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_-$ be the Cartan decomposition of $\bar{\mathfrak{g}}$, and let $(\bar{e}_+, \bar{h}, \bar{e}_-)$ be the Chevalley basis of $\bar{\mathfrak{g}}$, so $\bar{\mathfrak{n}}_\pm = \mathbb{C}e_\pm$ and $\bar{\mathfrak{h}} = \mathbb{C}\bar{h}$. Let $\langle\ ,\ \rangle_{\bar{\mathfrak{g}}}$ be the invariant symmetric bilinear form on $\bar{\mathfrak{g}}$ such that $\langle \bar{h}, \bar{h}\rangle_{\bar{\mathfrak{g}}} = 1$. Set $\mathfrak{g} = \bar{\mathfrak{g}} \times \bar{\mathfrak{g}}$, $\langle (x,y),(x',y')\rangle_{\mathfrak{g}} = \langle x, x'\rangle_{\bar{\mathfrak{g}}} - \langle y, y'\rangle_{\bar{\mathfrak{g}}}$. Set $\mathfrak{g}_+ = \{(x,x), x \in \bar{\mathfrak{g}}\}$ and $\mathfrak{g}_- = \{(\eta + \xi_+, -\eta + \xi_-), \xi_\pm \in \bar{\mathfrak{n}}_\pm, \eta \in \bar{\mathfrak{h}}\}$. Then $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. Let $(e_+, h, e_-, e_+^*, h^*, e_-^*)$ be the basis of $\mathfrak{g}$, such that $x = (\bar{x}, \bar{x})$ for $x \in \{e_+, h, e_-\}$, $e_+^* = (\bar{e}_+, 0)$, $h^* = (\bar{h}, -\bar{h})$ and $e_-^* = (0, \bar{e}_-)$.

A quantization of the Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is the algebra A with generators again denoted $(e_+, h, e_-, e_+^*, h^*, e_-^*)$, and relations

$$[h, e_\pm] = \pm 2e_\pm,\ [h^*, e_\pm^*] = 2e_\pm^*,\ [h, e_\pm^*] = \pm 2e_\pm^*,\ [h^*, e_\pm] = -2(e_\pm - 2e_\pm^*),$$
$$[e_+, e_-] = \frac{q^h - q^{-h}}{q - q^{-1}},\ [e_+, e_-^*] = \frac{q^h - q^{h^*}}{q - q^{-1}},\ [e_+^*, e_-] = \frac{q^{h^*} - q^{-h}}{q - q^{-1}},\ [e_+^*, e_-^*] = 0,$$
$$[h, h^*] = 0, \quad e_\pm^* e_\pm - q^{-2} e_\pm e_\pm^* = (1 - q^{-2})(e_\pm^*)^2,$$

where we set $q = \exp(\hbar)$. We define A_+ (resp., A_-) to be the subalgebra of A generated by e_+, h, e_- (resp., e_+^*, h^*, e_-^*). There is a unique algebra map $\Delta : A \to A \otimes A$, such that

$$\Delta(e_+) = e_+ \otimes q^h + 1 \otimes e_+, \Delta(e_-) = e_- \otimes 1 + q^{-h} \otimes e_-, \Delta(h) = h \otimes 1 + 1 \otimes h,$$
$$\Delta(e_+^*) = (e_+^* \otimes q^{h^*}) \left(1 - q^{-1}(q - q^{-1})^2 e_+^* \otimes e_-^*\right)^{-1} + 1 \otimes e_+^*,$$
$$\Delta(e_-^*) = e_-^* \otimes 1 + (q^{h^*} \otimes e_-^*) \left(1 - q^{-1}(q - q^{-1})^2 e_+^* \otimes e_-^*\right)^{-1},$$
$$\Delta(q^{h^*}) = (q^{h^*} \otimes q^{h^*}) \left(1 - q^{-1}(q - q^{-1})^2 e_+^* \otimes e_-^*\right)^{-1} \times$$
$$\times \left(1 - q^{-3}(q - q^{-1})^2 e_+^* \otimes e_-^*\right)^{-1}.$$

Then A_+ and A_- are Hopf subalgebras of A, and (A, A_+, A_-, Δ) is a quantization of the Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$. The algebra A is the Drinfeld double of (A_+, Δ) and its R-matrix is

$$\mathscr{R} = \exp_{q^2}\left(-(q - q^{-1}) e_-^* \otimes e_+\right) q^{\frac{1}{2} h^* \otimes h} \exp_{q^2}\left(-(q - q^{-1}) e_+^* \otimes e_-\right)^{-1},$$

where $\exp_{q^2}(z) = \sum_{n \geq 0} \frac{z^n}{[n]!}$ and $[n]! = \prod_{k=1}^n (1 + q^2 + \cdots + q^{2k-2})$.

Let us fix $\alpha \in \mathbb{C}$ and define $\mathfrak{h}_\alpha$ to be the subalgebra $\mathrm{Ad}(e^{\alpha h^*})(\mathfrak{g}_+)$ of $\mathfrak{g}$. The linear space $\mathfrak{h}_\alpha$ is spanned by h and $e_\pm + \beta e_\pm^*$, where $\beta = e^{4\alpha} - 1$. When $\beta \neq 0$, $(\mathfrak{g}, \mathfrak{g}_\pm, \mathfrak{g}_\mp, \mathfrak{h}_\alpha)$ are nongraded Manin quadruples. We now assume $\beta \neq 0$.

Here are quantizations of these Manin quadruples. Let us define B_α to be the subalgebra of A generated by h and $e_\pm + \beta e_\pm^*$. Let us set

$$F_\alpha = \Psi_\alpha(e_+^* \otimes e_-^*), \text{ with } \Psi_\alpha(z) = \frac{\exp_{q^2}\left(-(q - q^{-1}) e^{-4\alpha} z\right)}{\exp_{q^2}\left(-(q - q^{-1}) z\right)}.$$

Proposition 4.1 *$((A, \Delta, \mathscr{R}), A_\pm, B_\alpha, F_\alpha)$ are quantizations of the Manin quadruples $(\mathfrak{g}, \mathfrak{g}_\pm, \mathfrak{g}_\mp, \mathfrak{h}_\alpha)$.*

Proof. We have $B_\alpha \cap A_+ = \mathbb{C}[h][[\hbar]]$, and $B_\alpha \cap A_- = \mathbb{C}[[\hbar]]$. Let us set $u_\alpha = e^{\alpha h^*}$. Then $\Delta(u_\alpha) = F^*(u_\alpha \otimes u_\alpha)(F^*)^{-1}$, where $F^* = \exp_{q^2}(-(q - q^{-1}) e_+^* \otimes e_-^*)$. It follows that $F_\alpha = (u_\alpha \otimes u_\alpha)\Delta(u_\alpha)^{-1}$, which implies that F_α satisfies the cocycle identity.

Moreover, $B_\alpha = u_\alpha A_+ u_\alpha^{-1}$, therefore $F_\alpha \Delta(B_\alpha) F_\alpha^{-1} \subset B_\alpha^{\otimes 2}$. In fact, A equipped with the twisted coproduct $F_\alpha \Delta F_\alpha^{-1}$ is a Hopf algebra, and thus B_α is a Hopf subalgebra of $(A, F_\alpha \Delta F_\alpha^{-1})$.

Let us now show that F_α satisfies both condition (4), and the similar condition where A_+ is replaced by A_-. This follows from the conjunction of

$$\Delta(u_\alpha) \in \left(1 + \hbar (A_-)_0 \otimes (A_+)_0\right) (u_\alpha \otimes u_\alpha) \left(1 + \hbar A_0 \otimes (A_+)_0\right) \tag{5}$$

and

$$\Delta(u_\alpha) \in \left(1 + \hbar (A_+)_0 \otimes (A_-)_0\right) (u_\alpha \otimes u_\alpha) \left(1 + \hbar (A_+)_0 \otimes A_0\right). \tag{6}$$

Then

$$\Delta(u_\alpha) = F'(u_\alpha \otimes u_\alpha)(F')^{-1} = F''(u_\alpha \otimes u_\alpha)(F'')^{-1}, \tag{7}$$

where

$$F' = \exp_{q^2}(-(q-q^{-1}) e_+^* \otimes e_-), \quad F'' = \exp_{q^2}(-(q-q^{-1}) e_+ \otimes e_-^*).$$

The first equality of (7) proves (5), and the second one proves (6). ■

One may expect that the algebra-module over (A_+^{opp}, Δ) constructed by means of $((A, \Delta, \mathscr{R}), A_+, B_\alpha, F_\alpha)$ in Theorem 3.1 is a formal completion of the function algebra over a Podleś sphere ([17]).

Remark 11 One shows that for $x \in A_-$, $\Delta(x) = F^* \widetilde{\Delta}(x) (F^*)^{-1}$, where $\widetilde{\Delta}$ is the coproduct on A_- defined by $\widetilde{\Delta}(e_+^*) = e_+^* \otimes q^{h^*} + 1 \otimes e_+^*$, $\widetilde{\Delta}(q^{h^*}) = q^{h^*} \otimes q^{h^*}$, and $\widetilde{\Delta}(e_-^*) = e_-^* \otimes 1 + q^{h^*} \otimes e_-^*$. The completion of the Hopf algebra (A_-, Δ) with respect to the topology defined by its augmentation ideal should be isomorphic to the formal completion at the identity of the quantum coordinate ring of SL_2. The Hopf algebra A was obtained by a method similar to that of Drinfeld's "new realizations".

4.2. The case of loop algebras

Let us return to the situation of the Manin quadruple (1), in the case where $\mathscr{K}$ is a field of Laurent series and R is a ring of functions on an affine curve. In this situation, one can consider the following problems.

1) If the double quotient $G_+ \setminus G/H$ is equipped with the zero Poisson structure, the projection $G/H \to G_+ \setminus G/H$ is Poisson. The ring of formal functions on this double quotient is $\mathscr{O}_{G_+ \setminus G/H} = (U\mathfrak{g}/((U\mathfrak{g})\mathfrak{h} + \mathfrak{g}_+ U\mathfrak{g}))^*$. On the other hand, $(A^*)^{A_+, B} = \{\ell \in A^* | \forall a_+ \in A_+, b \in B, \ell(a_+ ab) = \ell(a)\varepsilon(a_+)\varepsilon(b)\}$ is a subalgebra of $(A^*)^B$. It is commutative because the R-matrix $\mathscr{R}$ of (A, Δ) belongs to $(A_+ \otimes A_-)_0^\times$ and the twisted R-matrix $F^{(21)} \mathscr{R} F^{-1}$ belongs to $(B \otimes A)_0^\times$ (see [9]).

It would be interesting to describe the algebra inclusion $(A^*)^{A_+, B} \subset (A^*)^B$, to see whether it is a flat deformation of $\mathscr{O}_{G_+ \setminus G/H}$, and when the level is critical, to describe the action of the quantum Sugawara field by commuting operators on $(A^*)^{A_+,B}$ and $(A^*)^B$. These operators could be related to the operators constructed in [10].

2) For g fixed in G, let ${}^g x$ denote the conjugate gxg^{-1} of an element x in G by g. One would like to describe the Poisson homogeneous spaces $G_\pm/(G_\pm \cap {}^g H)$, and to obtain quantizations of the Manin quadruples $(\mathfrak{g}, \mathfrak{g}_\pm, \mathfrak{g}_\mp, {}^g\mathfrak{h})$. A natural idea would be to start from the quantization of the Manin quadruple $(\mathfrak{g}, \mathfrak{g}_\pm, \mathfrak{g}_\mp, \mathfrak{h})$ obtained in [9], and to apply to B a suitable automorphism of A which lifts the automorphism $\mathrm{Ad}(g)$ of $U\mathfrak{g}$.

We hope to return to these questions elsewhere.

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Gauged q-fields

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Abstract. The straightforward description of q-deformed systems leads to transition amplitudes that are not numerically valued. To give physical meaning to these expressions without introducing *ad hoc* remedies, one may exploit an "internal" Fock space already defined by the q-algebra. This internal space may be interpreted in terms of internal degrees of freedom of the deformed system or alternatively in terms of non-locality. It is shown that the q-deformation may give stringy characteristics to a Yang-Mills theory.

Keywords: q-fields

Mathematics Subject Classifications (2000): 81R50, 81S05, 81P99, 81T10

To Moshé Flato

1. Introduction

Corresponding to the well known systems of quantum mechanics such as the harmonic oscillator or the hydrogen atom there are q-systems obtained by going over to quantum groups. The q-systems have more degrees of freedom than the system from which they are derived. When the standard field theories are similarly deformed the new degrees of freedom may be interpreted as expressions of non-locality exhibited by extended particles. Here we explore one aspect of this non-locality as it might appear in q-Yang-Mills theories.

2. The q-Yang-Mills theory

Let $\psi(x)$ be the basis of the fundamental representation of a gauge group. Then if $\psi'(x) = T(x)\psi(x)$, the covariant derivative ∇_μ transforms as

$$\nabla'_\mu = T\nabla_\mu T^{-1} \tag{2.1}$$

so that $\nabla'_\mu \psi' = T(\nabla_\mu \psi)$.

The corresponding gauge field A_μ is defined by $A_\mu = \nabla_\mu - \partial_\mu$. Then by (2.1)

$$A'_\mu = TA_\mu T^{-1} + T\partial_\mu T^{-1}.$$

The field strength is $F_{\mu\nu} = (\nabla_\mu, \nabla_\nu)$ and transforms as $F'_{\mu\nu} = TF_{\mu\nu}T^{-1}$ by (2.1). The nature of the gauge group is so far unspecified.

We now assume that $T \in SU_q(2)$. Then

$$T^t \varepsilon T = T\varepsilon T^t = \varepsilon \tag{2.2}$$

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where

$$\varepsilon = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}.$$

Then, if ψ is also a Dirac field, $\psi^t C\varepsilon\gamma^\mu \nabla_\mu \psi$ is invariant where ψ^t is the transpose of ψ while C and ε satisfy

$$\begin{aligned} L^t C L &= C \\ T^t \varepsilon T &= \varepsilon \end{aligned}$$

where L is a Lorentz transformation and C is the charge conjugation matrix. The basic Lagrangian may be chosen to be

$$S = \int d^4x \left[Q_\ell F_{\mu\nu} F^{\mu\nu} Q_r + i\psi^t C\varepsilon\gamma^\mu \nabla_\mu \psi + \frac{1}{2}[(\nabla_\mu \varphi)^t \varepsilon \nabla^\mu \varphi + \varphi^t \varepsilon \varphi] \right], \qquad (2.3)$$

where $Q_\ell' = Q_\ell T^{-1}$ and $Q_r' = TQ_r$, and where kinetic terms in Q_ℓ and Q_r are omitted. Here φ is a Lorentz scalar that is also a fundamental representation of $SU_q(2)$.

3. The $SU_q(2)$ algebra

Let

$$T(0) = \begin{pmatrix} \alpha_0 & \beta_0 \\ -q^{-1}\bar{\beta}_0 & \bar{\alpha}_0 \end{pmatrix}.$$

Then (2.2) implies

$$\begin{aligned} \alpha_0\beta_0 &= q\beta_0\alpha_0 & \alpha_0\bar{\alpha}_0 + \beta_0\bar{\beta}_0 &= 1 \\ \alpha_0\bar{\beta}_0 &= q\bar{\beta}_0\alpha_0 & \bar{\alpha}_0\alpha_0 + q_1^2\bar{\beta}_0\beta_0 &= 1 \\ \beta_0\bar{\beta}_0 &= \bar{\beta}_0\beta_0 & q_1 &= q^{-1}. \end{aligned} \qquad (3.1)$$

Let us next introduce

$$T(x) = \begin{pmatrix} e^{i\varphi(x)}\alpha_0 & e^{i\varphi(x)}\beta_0 \\ -q_1 e^{-i\varphi(x)}\bar{\beta}_0 & e^{-i\varphi(x)}\bar{\alpha}_0 \end{pmatrix} = \begin{pmatrix} \alpha(x) & \beta(x) \\ -q_1\bar{\beta}(x) & \bar{\alpha}(x) \end{pmatrix}.$$

Then the elements of $T(0)$ and $T(x)$ satisfy the same relations (3.1). We shall write $(\alpha, \beta, \bar{\beta}, \bar{\alpha})$ for the space dependent matrix elements.

There is associated with the algebra (3.1) a state space. Define the ground state, $|0\rangle$, by $\alpha|0\rangle = 0$. Since β and $\bar{\beta}$ commute we may require $|0\rangle$ to be a common eigenstate of β and $\bar{\beta}$. Then

$$\begin{aligned} \beta|0\rangle &= b|0\rangle \\ \bar{\beta}|0\rangle &= \bar{b}|0\rangle \end{aligned}$$

Note

$$(\bar{\alpha}\alpha + q_1^2\beta\bar{\beta})|0\rangle = |0\rangle \text{ or } |b|^2 = q^2. \tag{3.2}$$

Define $|n\rangle = \lambda_n\bar{\alpha}^n|0\rangle$. Then

$$\begin{aligned}\beta|n\rangle &= \lambda_n\beta\bar{\alpha}^n|0\rangle \\ &= \lambda_n q^n\bar{\alpha}^n\beta|0\rangle\end{aligned}$$

by iterating (3.1). Then $\beta|n\rangle = q^n b|n\rangle$. Likewise $\bar{\beta}|n\rangle = q^n\bar{b}|n\rangle$. One also finds

$$\lambda_n = \prod_0^{n-1}(1-|b|^2q^{2s})^{-1/2}$$

Finally define the analogue of the Hamiltonian of the oscillator:

$$H = \frac{1}{2}(\alpha\bar{\alpha} + \bar{\alpha}\alpha).$$

By (3.1)

$$H = 1 - \frac{1}{2}(1+q_1^2)\bar{\beta}\beta.$$

Then

$$H|n\rangle = \left[1 - \frac{1}{2}(1+q^2)q^{2n}\right]|n\rangle$$

by (3.2). Here the $|n\rangle$ are eigenstates of $\beta, \bar{\beta}$ and H. The levels depend on q^{2n} rather than on n, i.e. they are arranged in geometric rather than arithmetic progression and in this respect resemble the $\langle n\rangle$ order of the q-oscillator.

4. An Internal Space [1]

The existence of the algebra (3.1) leads to novel features of q-systems, as one sees in the q-deformations of familiar elementary systems such as the harmonic oscillator and the hydrogen atom.

In these examples one finds that the wave functions are not numerically valued, but lie in the q-algebra. To interpret these results according to the usual rules, however, it is necessary to calculate numerically valued transition probabilities between states represented by these wave functions. There is a natural procedure for doing this by utilizing a Fock space associated with the algebra (3.1). If such a path is followed, however, these simple one-particle problems like the oscillator and the hydrogen atom are endowed with internal degrees of freedom or partially promoted to the complexity of quantum field problems. In a similar way a quantum field-theoretic problem will acquire a second Fock space or internal degrees of freedom that can be given a non-local interpretation. We shall now see how this might work out for any q-deformed field theory, including Yang-Mills.

A general field lying in the q-algebra will have the following expansion

$$\psi_\mu(x) = \sum_\rho [f_\mu(\rho,x)a(\rho) + g_\mu(\rho,x)\bar{b}(\rho)]$$

where

$$\rho = (\vec{p}, r, s)$$
$$\sum_\rho = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d\vec{p}}{(2p_o)^{1/2}} \sum_{r,s}$$
$$f_\mu(\rho,x) = \sum_s U_\mu(\vec{p},r,s)e^{-ipx}\tau_s$$
$$g_\mu(\rho,x) = \sum_s V_\mu(\vec{p},r,s)e^{ipx}\tau_s$$

Here μ is a generic tensor index, $\bar{a}$ and $\bar{b}$ are creation operators for particles and antiparticles, the r-sum is the sum over different polarizations and the s-sum is the sum over generators of the q-algebra.

We may set

$$\psi_\mu = \sum_{s=1}^{4} \psi_{\mu s}\tau_s$$

where the τ_s may be chosen as follows:

$$\tau_s \equiv \tau_{ss'} = T_{jk}\delta_{s,j}\delta_{s',k}.$$

This choice of the τ_s implies a privileged gauge. Any other linear combination is allowed. A general gauge transformation will carry ψ into a space where the basis elements are no longer linear in τ_s.

5. Energy and mass of a *q*-scalar particle

Let us consider the contribution of the scalar field in (2.3). Then the energy operator is

$$\mathscr{E} = \int T_{00}d\vec{x} = \frac{1}{2}\int [(\nabla\varphi)^t\varepsilon\nabla\varphi + m_0^2\varphi^t\varepsilon\psi]d\vec{x} \tag{5.1}$$

where

$$\varphi = \left(\frac{1}{2\pi}\right)^{3/2} \int \frac{d\vec{p}}{(2p_0)^{1/2}} \sum_s U(p,s)\left[e^{-ipx}a(p,s) + e^{ipx}\bar{a}(p,s)\right]\tau_s \tag{5.2}$$

with the usual commutators for the a and $\bar{a}$. Here the s-sum is over the generators of the q-gauge group. We shall compute just the contribution of the free field, i.e. without the vector interaction.

Denote the basis states in Fock space by $|Nn\rangle$. Then

$$\bar{a}(\vec{p},s)a(\vec{p},s')\beta|N(\vec{p},s)n\rangle = N(\vec{p},s)q^n b|N(\vec{p},s)n\rangle\delta(s,s') \tag{5.3}$$

where $N(\vec{p},s)$ is the population number of the state $(\vec{p},s)$.

Now substitute (5.2) in (5.1) and use (5.3). Then the eigenvalue of $\mathscr{E}$ is given by

$$\mathscr{E}|Nn\rangle = \int d\vec{p}\sum_s U(\vec{p}.s)^t \varepsilon U(\vec{p},s)p_o N(\vec{p},s)\tau_s^2|Nn\rangle, \tag{5.4}$$

where $p_o^2 = \vec{p}^{\,2} + m^2$. Choose $\tau_s = \delta_{s1}\tau_1$, where $\tau_1 = \beta$. By (5.4)

$$\mathscr{E}|N(\vec{p},1)n\rangle = \int d\vec{p}\left[U(\vec{p},1)^t \varepsilon U(\vec{p},1)\right] p_o N(\vec{p},1)\beta^2|N(\vec{p},1)n\rangle.$$

The factor $[U(p,1)^t \varepsilon U(p,1)]$ is numerical and may be normalized to unity. Then the eigenvalues of the partial energy contributed by the free scalar field are

$$\mathscr{E}' = \int d\vec{p}N(\vec{p},1)p_o q^{2n+1}.$$

If q is near unity, say $1 + \frac{\varepsilon}{2}$, then the mass is

$$\begin{aligned} m_o q^{2n+1} &= m_o\left(1 + \frac{\varepsilon}{2}\right)^{2n+1} \\ &= m_o\left(1 + \frac{1}{2}\varepsilon + n\varepsilon\right) \end{aligned}$$

for low n. It follows that a point particle with mass m_o in a $q = 1$ theory becomes a particle with a mass spectrum and internal degrees of freedom in a $q \neq 1$ theory. For small values of n the mass spectrum resembles a string spectrum with the tension determined by q and m_o. If $q < 1$ spectrum will be inverted and bounded. In the present $SU_q(2)$ example, $q < 1$.

One may construct similar arguments for the mass terms of other q-fields. In every case, however, the actual dependence of the field on the q-algebra is gauge dependent. To make sense of this procedure one requires the introduction of a privileged gauge, but that is also the way in which the Higgs mechanism and the symmetry breaking vacuum completes the original Yang-Mills theory.

Alternative to introducing an internal state space is averaging over the q-group space with the aid of the Woronowicz integral [2]. In this alternative formulation one replaces the action (2.3) which lies in the q-algebra, and is therefore not numerically valued, by the following average over the algebra:

$$S = h\int d^4x\, L$$

where h stands for the Haar measure, or the Woronowicz integral, which is a linear functional that may be evaluated term by term according to

$$h[\alpha^s\beta^n\bar{\beta}^m] = \delta^{s0}\delta^{mn}\frac{q^n}{[m+1]_q}.$$

Thus h projects out a power of $\beta\bar{\beta}$ that is evaluated as the same power of q. This result can be easily compared with the very similar result obtained by the alternative procedure that employs the state space.

The apparent similarity of a local gauge theory based on $SU_q(2)$ to one based on $SU(2)$ is misleading, however, since only the latter has closure, i.e. if T_1 and $T_2 \in SU(2)$ then $T_1T_2 \in SU(2)$; but if T_1 and $T_2 \in SU_q(2)$ then $T_1T_2 \notin SU_q(2)$ unless the elements of T_1 commute with the elements of T_2. In other words T_1 and T_2 must belong to different copies of the algebra. For example, it is possible that $T_1 = T(x_1)$ and $T_2 = T(x_2)$ where x_1 and x_2 both lie in spacetime but are not causally related; another possibility is offered by a Kaluza-Klein theory if x_1 and x_2 do not both lie in 4-dimensional spacetime. Such a collection of transformations may be regarded as a non-local group.

Alternatively one can assume that there are only two "gauges" and that T is a duality transformation. One example of this kind of duality is realized by the q-harmonic oscillator which has equivalent descriptions in terms of either (x,p) or $(a,\bar{a})$. These two representations are related by the q-canonical transformation:

$$\begin{pmatrix} a \\ \bar{a} \end{pmatrix} = T \begin{pmatrix} \frac{i}{\hbar}p \\ x \end{pmatrix} \qquad T \in SU_q(2).$$

Both sets satisfy q-commutation rules, namely:

$$\begin{aligned} a\bar{a} - q\bar{a}a &= 1, \\ qxp - px &= i\hbar. \end{aligned}$$

One may promote the oscillator to the role of a scalar field by introducing the conjugate fields $\pi(x)$ and $\psi(x)$ satisfying

$$q\psi(x')\pi(x) - \pi(x)\psi(x') = i\hbar\delta(x-x').$$

If the Fourier components of $\pi(x)$ and $\psi(x)$ are p_k and q_k and if we set

$$\begin{pmatrix} a_k \\ \bar{a}_k \end{pmatrix} = T \begin{pmatrix} \frac{i}{\hbar}p_k \\ q_k \end{pmatrix}$$

then the field oscillators will satisfy q-commutators:

$$a_k\bar{a}_{k'} - q\bar{a}_{k'}a_k = \delta_{kk'}. \tag{5.5}$$

If one now calculates the energy of the free field one finds the eigenvalues

$$\langle n_k \rangle \hbar\omega_k$$

instead of

$$n_k\hbar\omega_k.$$

If q is close to unity and we set $q = 1 + \varepsilon$, then

$$\begin{aligned} \langle n \rangle &= \frac{q^n - 1}{q - 1} \\ \frac{\langle n \rangle}{n} &= 1 + \frac{n-1}{2}\ \varepsilon \end{aligned}$$

Hence

$$\langle n \rangle \hbar\omega = n\left[\hbar\omega + \frac{n-1}{2}\varepsilon\hbar\omega\right].$$

This result may be interpreted to describe either a field with populations of normal modes increasing as $\langle n \rangle$ with fixed mass or as populations increasing linearly with n but with increasing mass. This example differs from the preceding case in that here there is no "internal" quantum number and in addition the field commutators were normal in the other case. Both examples illustrate how the introduction of the q-algebra may alter the particle spectrum.

To complete the discussion of this case it is again necessary to introduce the internal space in order to compute transition probabilities. Then the field acquires an internal mass spectrum as in the preceding example. In both cases the internal $(\alpha, \bar{\alpha})$ space is associated with the algebra (3.1). In the present case the $(\alpha, \bar{\alpha})$ space results from (5.5). In the previous (q-Yang-Mills) case the dynamical fields lie in the algebra (3.1) because of the postulated $SU_q(2)$ invariance.

6. Transition amplitudes

We may consider the emission of a q-vector by a q-spinor induced by the following interaction appearing in (2.3):

$$\int i\psi^t(x) C\varepsilon\gamma^\mu A_\mu(x)\psi(x) d\vec{x}$$

where

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{(2p_o)^{1/2}} \sum_p e_\mu(\rho)\left[e^{-ipx}A(\rho) + e^{ipx}\bar{a}(\rho)\right]$$

and

$$e_\mu(\rho) = \sum_s e_\mu(\rho, s)\tau_s \qquad \rho = (\vec{p}, r).$$

Here $(\vec{p}, r)$ labels momentum and polarization of the vector particle while τ_s is to be summed over a selected portion of the internal algebra generated by $(\alpha, \bar{\alpha}, \beta, \bar{\beta})$. Similarly we set

$$\psi(x) = \sum_s \psi_s(x)\hat{\tau}_s.$$

Then the general matrix element of interest is

$$\int d\vec{x} \langle N'_\psi N'_A n'_\psi n'_A | i\psi^t(x) C\varepsilon\gamma^\mu A_\mu(x)\psi(x) | N_\psi N_A n_\psi n_A \rangle$$

where N and n refer to the external and internal Fock spaces.

The emission of a specific vector particle (ρ) by the fermionic source will increase the corresponding population of the field from N to $N+1$. If a single fermion makes a transition from n_ψ to n'_ψ in the same event then the amplitude for the joint transition is

$$\int d\vec{x} \langle n'_\psi n'_A | i\psi^t(x) C\varepsilon\gamma^\mu \psi(x) A_\mu(x) | n_\psi n_A \rangle (N_A+1)^{1/2}$$

or

$$\int d\vec{x} \langle n'_\psi | i\psi^t(x) C\varepsilon\gamma^\mu \psi(x) | n_\psi \rangle \langle n'_A | A_\mu(x) | n_A \rangle (N_A+1)^{1/2}.$$

In the $q=1$ theories there is no need of internal quantum numbers like n_ψ and n_A.

Note that if τ_s or $\hat{\tau}_s$ is a function of β and $\bar{\beta}$ and does not contain free factors of α and $\bar{\alpha}$, then there is no change in internal quantum number in the transition. On the other hand one has

$$\langle n' | \bar{\alpha} | n \rangle = \delta(n', n+1)(1-q^{2n+2})^{1/2}.$$

The ratio of the emission to the absorption probabilities of a vector particle is

$$R = \frac{|\langle n+1|\bar{\alpha}|n\rangle|^2}{|\langle n-1|\alpha|n\rangle|^2} = \frac{1-q^{2(n+1)}}{1-q^{2n}}$$

multiplied by the corresponding factor for the spinor transition.

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Generalized coherent states for arbitrary quantum systems

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Abstract. Recent developments [3] in the construction of coherent states for an arbitrary quantum system with Hamiltonian H are presented here. These CS families $\{\mid J,\gamma\rangle\}$ are required to solve the unity, to be temporally stable, $e^{-iHt/\hbar}\mid J,\gamma\rangle = \mid J,\gamma+\omega t\rangle$, and the J functional dependence of the energy mean value $\langle J,\gamma \mid H \mid J,\gamma\rangle = E(J)$ is assigned possibly along lines which are classically meaningful. Further extensions are finally proposed.

Keywords: coherent states, quantization, wave packets, revival structures

Mathematics Subject Classifications (2000): 81R30, 81S99

1. A review of standard CS properties

Since (standard) Coherent States were invented first by Schrödinger and rediscovered by Klauder [1], Glauber, Sudarshan, many generalizations have been proposed. Each of them retain some of the properties of the standard ones. Let us recall the familiar expression of the latter as infinite superposition of harmonic oscillator eigenstates,

$$\mid z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\mid n\rangle, \tag{1.1}$$

and let us give a brief review of those properties which on their whole own give these states a strong status of uniqueness.

P$_0$ *The map* $\mathbb{C}\ni z \rightarrow \mid z\rangle \in \mathrm{L}^2(\mathbb{R})$ *is continuous.*

P$_1$ $\mid z\rangle$ *is annihilation eigenvector:* $a\mid z\rangle = z\mid z\rangle$.

P$_2$ *The CS family resolves the unity:* $\frac{1}{\pi}\int_{\mathbb{C}}\mid z\rangle\langle z\mid d^2z = \mathrm{I}$.

P$_3$ *The CS saturate Heisenberg inequality:* $\Delta Q\,\Delta P = 1/2$.

P$_4$ *The CS family is temporally stable:* $e^{-i\mathscr{H}t}\mid z\rangle = \mid e^{-i\omega t}z\rangle$, *where* $\mathscr{H}$ *is the harmonic oscillator Hamiltonian. The latter is shifted such that* $\mathscr{H}\mid n\rangle = \omega n\mid n\rangle$.

P$_5$ *The mean value (or "lower symbol") of the Hamiltonian mimics the classical relation energy-action:* $\check{\mathscr{H}}(z) \equiv \langle z\mid \mathscr{H}\mid z\rangle = \omega|z|^2$.

P$_6$ *The CS family is the orbit of the ground state under the action of the Weyl-Heisenberg displacement operator:* $\mid z\rangle = e^{za^\dagger - \bar{z}a}\mid 0\rangle \equiv D(z)\mid 0\rangle$.

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P_7 *The Weyl-Heisenberg covariance follows from the above:*

$$\mathscr{U}(s,\zeta)\mid z\rangle = e^{i(s+\Im(\zeta\bar{z}))}\mid z+\zeta\rangle.$$

P_8 *The coherent states provide a straightforward quantization scheme:*

Classical state $z \to \mid z\rangle\langle z\mid$ *quantum state.*

Thus, these properties cover a wide spectrum, starting from a "wave-packet" side (Equation (1.1) and Properties P_3 and P_4) through an algebraic side (P_1), a group representation side (P_6 and P_7), a functional analysis side (P_2) to end with the ubiquitous problematic of the relationship between classical and quantum worlds (P_5 and P_8) [4], [5]. The generalisations we are going to describe here take place on the wave-packet side. They are in the continuation of recent papers [2, 3], and preserve most of the properties listed in the above, if not all of them, at the price of some adaptation, of course. For the group representation and functional analysis aspects, we refer to [6] with references therein.

2. Coherent states for discrete spectrum dynamics

Let us start like in [3] with a minimal set of requirements. Suppose we deal with a quantum system described by a positive Hamiltonian $\mathscr{H}$ with purely discrete nondegenerate spectrum,

$$\mathscr{H}\mid n\rangle = \omega e_n\mid n\rangle, \quad n \geq 0, \tag{2.1}$$

where the strictly increasing sequence $0 = e_0 < e_1 < e_2 < \cdots < e_n < \cdots$ is made of pure numbers ($\hbar = 1$) and can be finite or infinite. Let us denote by H the Hilbertian span of the orthonormal set $\{\mid n\rangle\}_{n\geq 0}$. We then consider the abstract **normalized** wave packet in H:

$$\mid J,\gamma\rangle_\rho = \frac{1}{\sqrt{\mathscr{N}_\rho(J)}}\sum_{n\geq 0}\frac{J^{n/2}}{\sqrt{\rho_n}}e^{-ie_n\gamma}\mid n\rangle, \tag{2.2}$$

where $\rho_0 = 1$, $J \geq 0$ and $-\infty < \gamma < \infty$. It is clear that we have here a formal generalization of (1.1) where $z = \sqrt{J}e^{-in\gamma}$, $\rho_n = n!$, and $\mathscr{N}_\rho(J) = e^{|z|^2}$. Let us now examine the validity of some of the standard CS properties in the present case.

Eigenvector property. Property P_1 formally holds with the lowering operator defined by

$$a_\rho\mid n\rangle = \sqrt{\rho_n/\rho_{n-1}}\mid n-1\rangle,\; e^{-i\mathscr{H}t}a_\rho e^{i\mathscr{H}t}\mid J,\gamma\rangle_\rho = \sqrt{J}\mid J,\gamma\rangle_\rho, \tag{2.3}$$

for $n \geq 1$, and $a_\rho\mid 0\rangle = 0$.

Temporal stability. Property P_4 holds by construction:

$$e^{-i\mathscr{H}t}\mid J,\gamma\rangle_\rho = \mid J,\gamma+\omega t\rangle_\rho. \tag{2.4}$$

Normalization. This property imposes the power series

$$\mathcal{N}_\rho(J) = \sum_{n\geq 0} \frac{J^n}{\rho_n} \tag{2.5}$$

to be non trivial, which means that the convergence radius

$$R = \overline{\lim}_{n\to\infty} \sqrt[n]{\rho_n} > J \geq 0$$

is not null. Continuity property P_0 immediately follows.

Resolution of the unity. This is the crucial point because we have to solve a classical moment problem. Indeed, $\rho_n > 0$ is assumed to be a probability distribution moment

$$\rho_n = \int_0^R J^n w_\rho(J)\, dJ,\; w_\rho(J) \geq 0. \tag{2.6}$$

Then Property P_2 holds in the following sense:

$$\begin{aligned} I &= \int \mid J,\gamma\rangle_{\rho\rho}\langle J,\gamma \mid d\mu_\rho(J,\gamma) \\ &\equiv \lim_{\Gamma\to\infty} \frac{1}{2\Gamma} \int_{-\Gamma}^{\Gamma} d\gamma \int_0^R \mathcal{N}_\rho(J) w_\rho(J)\, dJ \mid J,\gamma\rangle_{\rho\rho}\langle J,\gamma \mid . \end{aligned} \tag{2.7}$$

It often happens that the determination of the distribution w_ρ, with support in $[0,R]$, is quite intractable, while another one, say $w_{\rho'}$, with same support, and giving rise to the moments

$$\rho'_n = \int_0^R J^n w_{\rho'}(J)\, dJ,\; w_{\rho'}(J) \geq 0, \tag{2.8}$$

may be also relevant to our purposes. Of course ρ'_n should be equivalent to ρ_n in the sense that both series $\mathcal{N}_\rho(J)$ and $\mathcal{N}_{\rho'}(J)$ have same convergence radius. So let us see what happens when we replace w_ρ by $w_{\rho'}$ in the $d\mu_\rho$ of (2.7). We now get the resolution of a positive diagonal operator $\Delta \equiv \delta(N)$, $\Delta \mid n\rangle = \delta_n \mid n\rangle$,

$$\begin{aligned} \Delta &= \int \mid J,\gamma\rangle_{\rho\rho}\langle J,\gamma \mid d\mu'(J,\gamma) \\ &\Longleftrightarrow\; I = \int (\Delta)^{-1/2} \mid J,\gamma\rangle_{\rho\rho}\langle J,\gamma \mid (\Delta)^{-1/2} d\mu'(J,\gamma), \end{aligned} \tag{2.9}$$

where $d\mu'(J,\gamma) = \mathcal{N}_\rho(J) w_{\rho'}(J)\, dJ\, d\Gamma$, and $\delta_n = \rho'_n/\rho_n$. This operator should be bounded with bounded inverse so that $\mathcal{N}_\rho(J)$ and $\mathcal{N}_{\rho'}(J)$ do have same convergence radius. The triple $(\Delta, \{\mid J,\gamma\rangle_\rho\}, d\mu')$ is called a *frame* [6].

3. Statistical and semi-classical aspects

At this point we should notice the remarkable interplay between two statistical aspects of the states (2.2):

Probability distribution $w_\rho(J)$

We could say this $w_\rho(J)$ pertains to the classical (statistical) side of the states $| J,\gamma\rangle_\rho$ if we take the freedom to give the measure $d\mu_\rho(J,\gamma)$ in (2.7) a classical phase space interpretation, which would be perfectly allowable in the standard case. As a matter of fact, let us associate to a quantum observable A its (non-unique) components $\hat{A}(J,\gamma)$ as seen from the continuous "coherent frame" $\{| J,\gamma\rangle_{\rho\rho}\langle J,\gamma |\}$:

$$A = \int \hat{A}(J,\gamma) \,| J,\gamma\rangle_{\rho\rho}\langle J,\gamma |\, d\mu_\rho(J,\gamma). \tag{3.1}$$

The function $\hat{A}(J,\gamma)$, which exists for a large class of operators A, is also called upper (see Lieb in [5]) or contravariant (Berezin in [7]) symbol. In particular, for an observable $f(N)$ diagonal with respect to $\{| n\rangle\}$, we have

$$f(N) = \int \hat{f}(J) \,| J,\gamma\rangle_{\rho\rho}\langle J,\gamma |\, d\mu_\rho(J,\gamma),$$
$$\text{which implies } f(n) = \frac{1}{\rho_n}\int_0^R \hat{f}(J) J^n w_\rho(J)\, dJ. \tag{3.2}$$

Wave-packet weighting probability

Alternatively, the moment ρ_n is part of the purely quantal weighting probability

$$|\langle n \mid J,\gamma\rangle_\rho|^2 \equiv |c_n|^2 = \frac{1}{\mathcal{N}_\rho(J)}\frac{J^n}{\rho_n}. \tag{3.3}$$

Let us now associate to the observable A its lower (or covariant) symbol which is nothing but its mean value in the state $| J,\gamma\rangle_\rho$:

$$\check{A}(J,\gamma) = {}_\rho\langle J,\gamma \mid A \mid J,\gamma\rangle_\rho. \tag{3.4}$$

In particular, we have for the diagonal observable $f(N)$:

$$\check{f}(J) = \sum_{n\geq 0} |c_n|^2 f(n) \equiv \langle f(n)\rangle. \tag{3.5}$$

Therefore, the weighting probability determines lower symbols of diagonal observables. It also determines the semi-classical revival structure (Averbukh and Perelman [8]) of the coherent state $| J,\gamma\rangle_\rho$. Let us briefly recall what we mean by this temporal concept of revival and refer to the comprehensive review of Bluhm, Kostelecký, and Porter [9] for more details. Suppose the expansion (2.2) is strongly weighted around a mean value $\check{N} = \langle n\rangle$ for the number operator N. For instance, in the case of the harmonic oscillator (1.1), the distribution $|c_n|^2 = e^{-|z|^2}|z|^2/n!$ is Poissonian and is weighted around $\check{N} = |z|^2$ with spread $\sigma \approx |z| = \sqrt{\check{N}}$. In the general case (2.2), let us choose $\bar{n} \in \mathbb{N}$ as being the closest integer to $\check{N}$ for a given J. With the hypothesis that the spread $\sigma \approx \Delta N$ is small compared with $\check{N} \approx \bar{n}$, what we should expect in general for large J (semi-classical behavior!), we expand the energy $E_n = \omega e_n$ in a Taylor series in n around the centrally excited value $\bar{n}$:

$$E_n \simeq E_{\bar{n}} + E'_{\bar{n}}(n-\bar{n}) + \frac{1}{2}E''_{\bar{n}}(n-\bar{n})^2 + \frac{1}{6}E'''_{\bar{n}}(n-\bar{n})^3 + \cdots, \tag{3.6}$$

where each prime on $E_{\bar{n}}$ denotes a derivative. These derivatives define distinct time scales: the classical period $T_{\rm cl} = 2\pi/|E'_{\bar{n}}|$, the revival time $t_{\rm rev} = 2\pi/\frac{1}{2}|E''_{\bar{n}}|$, the super-revival time $t_{\rm sr} = 2\pi/\frac{1}{6}|E'''_{\bar{n}}|$, and so on. Reporting this expansion in the evolution factor $e^{-i\gamma E_n/\omega}$ of (2.2) allows us to understand the possible occurrence of a quasiperiodic revival structure of these coherent states at high J, which would be well illustrated by the behavior in function of the evolution parameter γ of the absolute square of the so-called autocorrelation function of the coherent states:

$$|{}_\rho\langle J,0 \mid J,\gamma\rangle_\rho|^2 = |{}_\rho\langle J,0 \mid e^{-i\mathscr{H}\gamma/\omega} \mid J,0\rangle_\rho|^2. \tag{3.7}$$

It is clear from (3.6) that the revival features will be more or less apparent according to the value of the deviation $(n-\bar{n})$ relatively to n. A quantitative estimate is afforded by the so-called Mandel parameter $Q(J)$ defined as follows.

$$Q(J) = \frac{(\Delta N)^2}{\check{N}} - 1 = J\frac{d}{dJ}\log\frac{d}{dJ}\log\mathscr{N}_\rho(J). \tag{3.8}$$

This parameter allows one to compare the weighting distribution $|c_n|^2$ with the Poisson distribution: The latter is

- Poissonian if $Q = 0$,
- super-Poissonian if $Q > 0$,
- sub-Poissonian if $Q < 0$.

We are now aware of the basic importance of the distribution $w_\rho(J)$ in regard to these statistical considerations.

4. Imposing Hamiltonian lower symbol

So far the only observable we have at our disposal is the Hamiltonian. Suppose there exists some presupposed functional form

$$\check{\mathscr{H}}(J) = {}_\rho\langle J,\gamma \mid \mathscr{H} \mid J,\gamma\rangle_\rho = \omega\sum_{n\geq 0}|c_n|^2 e_n = \omega\langle e_n\rangle, \tag{4.1}$$

for the lower symbol of the Hamiltonian. We could for instance consider as a prerequisite the classical energy-action dependence for systems with one degree of freedom,

$$J = \frac{1}{2\pi}\oint p(E = \check{\mathscr{H}})dq, \tag{4.2}$$

and this way deserves to be carefully explored in view of studying the (semi-) classical behavior of the corresponding quantum system. Suppose that $\check{\mathscr{H}}(J)$ has a series expansion in the neighborhood of $J = 0$.

$$\check{\mathscr{H}}(J) = \omega J^{\nu} \sum_{n\geq 0} \alpha_n J^n. \tag{4.3}$$

Then, by simple identification with the expansion

$$\mathscr{N}_{\rho}(J)\check{\mathscr{H}}(J) = \omega \sum_{n\geq 0} \frac{e_n}{\rho_n} J^n, \tag{4.4}$$

we obtain

$$\nu = 0, \quad \alpha_0 = 0, \quad \frac{e_n}{\rho_n} = \sum_{m=0}^{n} \alpha_{n-m} \frac{1}{\rho_m}, \quad \rho_0 = 1. \tag{4.5}$$

The solution ρ_n to Equation (4.5) can be written in terms of the Jackson "factorial" [10]

$$[e_n]! \overset{def}{=} e_1 e_2 \cdots e_n, \; [e_n]! = 1, \tag{4.6}$$

as follows:

$$\rho_n = \frac{[e_n]!}{p_n}, \quad \text{with} \tag{4.7}$$

$$p_n = \sum_{m=0}^{n} \frac{[e_{n-1}]!}{[e_m]!} \alpha_{n-m}, \quad p_0 = 1. \tag{4.8}$$

For instance,

$$\rho_1 = \frac{e_1}{\alpha_1}, \quad \rho_2 = \frac{e_1 e_2}{e_1 \alpha_2 + \alpha_1^2}, \quad \rho_3 = \frac{e_1 e_2 e_3}{e_1 \alpha_2 + (e_1 + e_2)\alpha_1 \alpha_2 + \alpha_1^3} \cdots$$

Let us give a few explicit examples.

- *The linear case*: $\check{\mathscr{H}}(J) = \omega J$. It is the simplest one, of course, and gives rise to the "Action-Angle" coherent states introduced in [3]. Here the ρ_n's are exactly the Jackson factorials:

$$\rho_n = [e_n]!. \tag{4.9}$$

- *The single power case*: $\check{\mathscr{H}}(J) = \omega J^q$, $q = 2, 3, \cdots$. Then the corresponding CS do not generate the whole Hilbert space H:

$$\mid J, \gamma\rangle_{\rho} = \frac{1}{\sqrt{\mathscr{N}_{\rho}(J)}} \sum_{n\geq 0} \frac{J^{nq/2}}{\sqrt{e_q e_{2q} \cdots e_{nq}}} e^{-ie_{nq}\gamma} \mid nq\rangle. \tag{4.10}$$

- *The geometric case*: $\check{\mathscr{H}}(J) = \omega/(1-J)$. Then the denominator p_n in (4.7) is given by

$$p_n = \sum_{k=1}^{n-1} \sum_{1\leq i_1<i_2<\cdots<i_k\leq n-1} e_{i_1} e_{i_2} \cdots e_{i_k}. \tag{4.11}$$

Let us end this section by noting that the reverse to the above approach is also of great interest, at least from a more "experimental" point of view. Suppose the coherent states $|\, J, \gamma\rangle_\rho$ are prepared in such a way that the weighting distribution $|c_n|^2 = \frac{1}{\mathcal{N}_\rho(J)} \frac{J^n}{\rho_n}$ (*e.g.* a Gaussian-like distribution) is determined by the experimental conditions. Then the coefficients of the lower symbol series (4.3) will be given by inverting Equation (4.5): $\alpha_0 = 0$, $\alpha_n = \sum_{m=0}^{n} c_m(\rho_1, \cdots, \rho_n) e_m$.

5. "Action-Angle" coherent states

Let us now concentrate on the linear case $\check{\mathscr{H}}(J) = \omega J$, which amounts to require Property P_5. We shall adopt here the simplified notations:

$$|\, J, \gamma\rangle = \frac{1}{\sqrt{\mathrm{E}(J)}} \sum_{n\geq 0} \frac{J^{n/2}}{\sqrt{[e_n]!}} e^{-ie_n\gamma} \,|\, n\rangle, \tag{5.1}$$

where $\mathrm{E}(J)$ is the Jackson generalized exponential

$$\mathrm{E}(J) = \sum_{n\geq 0} \frac{J^n}{[e_n]!}. \tag{5.2}$$

The algebraic structure exemplified by Property P_1 and (2.3) may now display nontrivial interesting features. Indeed, it is clear from the definition of lowering and raising operators

$$a \,|\, n\rangle = \sqrt{e_n} \,|\, n-1\rangle, \quad a^\dagger \,|\, n\rangle = \sqrt{e_{n+1}} \,|\, n+1\rangle, \tag{5.3}$$

that the Hamiltonian factorizes as

$$\mathscr{H} = \omega a^\dagger a \equiv \omega e(N), \quad e(N) \,|\, n\rangle = e_n \,|\, n\rangle. \tag{5.4}$$

Now given any lowering operator a, $a \,|\, 0\rangle = 0$, of the above type, and any diagonal (real) operator $f(N)$, $f(N) \,|\, n\rangle = f_n \,|\, n\rangle$, the following commutation rules hold true:

$$af(N) = f(N+1)a, \quad f(N)a^\dagger = a^\dagger f(N+1), \tag{5.5}$$

$$[a, f(N)] = f'(N)a = af'(N-1), \quad [a^\dagger, f(N)] = -a^\dagger f'(N) = -f'(N-1)a^\dagger, \tag{5.6}$$

where $f'(N) \equiv f(N+1) - f(N)$ is the finite difference derivative of $f(N)$. Of course, the sense of the above *any*'s should be mellowed in view of operator domain considerations. In the present case, we also have

$$[a, a^\dagger] = e'(N). \tag{5.7}$$

So we can think about the existence of an interesting "dynamical" or "spectrum generating" Lie algebra generated by $\{a, a^\dagger, e'(N), \cdots\}$, where the dots mean further generators coming out from iterating the above commutators [11]. This algebra is generically infinite. However, non trivial structures, like quadratic algebra, can occur provided that the operator function $e(N)$ is solution of some required finite difference equation. Work in this very rich classification scheme is in progress (Lafortune, Gazeau, and Winternitz [12]).

6. Two examples of action-angle coherent states

6.1. Coulomb-like spectrum

Our first example concerns a Coulomb-like discrete spectrum:

$$e_n = (1 - \frac{1}{(n+1)^2}), \text{ for which } \rho_n = [e_n]! = \frac{1}{2}\frac{n+2}{n+1}. \tag{6.1}$$

So the convergence radius of the generalized exponential is $R = 1$. This series can be given the closed form:

$$\mathrm{E}(J) = \frac{2}{J}(\frac{1}{1-J} + \frac{\log(1-J)}{J}). \tag{6.2}$$

The distribution probability $w(J)$ is explicitly given by:

$$\rho_n = \int_0^1 J^n w(J)\, dJ \ \text{ where } \ w(J) = \frac{1}{J}(1 + \delta(J - 1^-)). \tag{6.3}$$

On the other hand, the weighting distribution is given by

$$|c_n|^2 = \frac{n+1}{n+2}\frac{(1-J)J^{n+2}}{J + (1-J)\log(1-J)}. \tag{6.4}$$

This distribution is super-Poissonian. This is easily inferred from the estimate on the parameter [13]:

$$\sigma \equiv \frac{e_{n+1}/e_n}{(n+1)/n} < 1 \Rightarrow Q < 0. \tag{6.5}$$

At the moment we do not have any interesting result about the algebraic structure, let alone the defining finite difference equation for $e(N)$

$$e(N) = \exp(\log N)''.$$

6.2. Infinite square well and Pöschl-Teller potentials

The paper of Ref.[14] is devoted to the infinite square well and Pöschl-Teller potential example. So we just give here a brief account of this nice illustration of many features of the action-angle coherent states. We consider the Hamiltonian

$$\mathscr{H} = \frac{P^2}{2m} + V(Q) - \frac{\hbar^2}{8ma^2}(\lambda+\kappa)^2, \tag{6.6}$$

for a particle on the line and submitted to the potential

$$V(x) = \begin{cases} 0 & 0 < x < \pi a \\ \infty & x \leq 0,\ x \geq \pi a \end{cases}, \tag{6.7}$$

for $\lambda = 1 = \kappa$ (*infinite square well*),

$$V(x) = \begin{cases} \frac{\hbar^2}{8ma^2}\left(\frac{\lambda(\lambda-1)}{\cos^2 x/2a} + \frac{\kappa(\kappa-1)}{\sin^2 x/2a}\right) & 0 < x < \pi a \\ \infty & x \leq 0,\ x \geq \pi a \end{cases}, \tag{6.8}$$

for $\lambda > 1,\ \kappa > 1$ (*Pöschl-Teller potential*). Note that we have here reintroduced the Planck constant into the physical quantities expressions. In a certain sense, the Pöschl-Teller potential [15] interpolates between the harmonic oscillator and the infinite square well. We shall here treat the former and the latter on the same footing, since the infinite well takes place at the limit $\lambda = 1 = \kappa$. Eigenvalues and eigenvectors are given by:

$$\mathscr{H} \mid n\rangle = \hbar\omega e_n \mid n\rangle, \quad e_n = n(n+\lambda+\kappa), \quad \omega = \frac{\hbar}{2ma^2}, \tag{6.9}$$

$$\begin{aligned}\langle x \mid n\rangle =& k_n (\cos x/2a)^{\lambda} (\sin x/2a)^{\kappa}\,{}_2\times \\ & \mathrm{F}_1(-n, n+\lambda+\kappa; \kappa+1/2; \sin^2 x/2a).\end{aligned} \tag{6.10}$$

- *Algebraic side.* We put $e(N) = N(N+\lambda+\kappa)$. The Lie algebra generated by $\{a, a^{\dagger}, e'(N), \cdots\}$ is finite and is $su(1,1)$, as it actually holds for any algebra built from an $e(N)$ quadratic in N. Indeed,

  $$[a,a^{\dagger}] = e'(N) = 2N+1+\lambda+\kappa,\ [a,e'(N)] = 2a,\ [a^{\dagger},e'(N)] = -2a^{\dagger}. \tag{6.11}$$

  Note that $e''(N) = 2$, $e'''(N) = 0$. Therefore, we can identify the Hilbertian span of the $\mid n\rangle$'s via the values assumed by the $su(1,1)$ Casimir operator.

  $$\mathscr{C} = -2e'(N)(2e'(N)-1) + e(N) \equiv \eta(\eta-1)\mathrm{I} \Rightarrow \eta = \frac{\lambda+\kappa+1}{2} \geq \frac{3}{2}. \tag{6.12}$$

  This range of values falls into the discrete series representation interval $1/2 < \eta < \infty$ for $su(1,1)$ or, equivalently, for the universal covering of $SU(1,1)$. Note that $\eta = 3/2$ corresponds to the infinite well.

- *Coherent states.* They read in the present case:

$$| J,\gamma\rangle = \frac{1}{\sqrt{\mathrm{E}(J)}} \sum_{n\geq 0} \frac{J^{n/2}}{\sqrt{[e_n]!}} e^{-ie_n\gamma} \, | n\rangle, \tag{6.13}$$

$$\rho_n = [e_n]! = \frac{n!\Gamma(n+\nu+1)}{\Gamma(\nu+1)}, \ \nu \equiv \lambda+\kappa \geq 2, \tag{6.14}$$

$$\mathrm{E}(J) = \Gamma(\nu+1)\frac{\mathrm{I}_\nu(2\sqrt{J})}{J^{\nu/2}}, \ R = \infty, \tag{6.15}$$

where I_ν is a modified Bessel function . Note that the $\nu = 2$ case corresponds to the infinite well. At this point, we should remark that, for $\gamma = 0$ and for complex values of J, these coherent states are nothing else but the Barut-Girardello states [16]. However, the construction of the latter was motivated by purely mathematical considerations, more precisely by the aim to extend to the algebra $su(1,1)$ the eigenvector property P_1.

- *Resolution of the identity.*

$$\mathrm{I} = \lim_{\Gamma\to\infty} \frac{1}{2\Gamma} \int_{-\Gamma}^{\Gamma} d\gamma \int_0^\infty E(J) w(J)\, dJ \; | J,\gamma\rangle\langle J,\gamma \,|,$$

$$\text{with } \rho_n = \int_0^\infty J^n w(J)\, dJ, \ w(J) = \frac{2J^{\nu/2}}{\Gamma(\nu+1)} \mathrm{K}_\nu(2\sqrt{J}), \tag{6.16}$$

where K_ν is a modified Bessel function.

- *Infinite square well.* Let us end this section by some considerations on the specific case of the infinite well [14], since it is the simplest one and should not qualitatively differ from the Pöschl-Teller cases. The weighting probability $|c_n|^2$ is sub-Poissonian and reads

$$|c_n|^2 = \frac{1}{n!(n+2)!} \frac{J^{n+1}}{\mathrm{I}_2(2\sqrt{J})}. \tag{6.17}$$

The revival structure is encoded by the expansion

$$E_n = \frac{\hbar^2}{2ma^2} n(n+2) = \frac{\hbar^2}{2ma^2}[\bar{n}(\bar{n}+2) + 2(\bar{n}+1)(n-\bar{n}) + (n-\bar{n})^2], \tag{6.18}$$

and so the two characteristic times are the classical period $T_{\mathrm{cl}} = 2\pi ma^2/\hbar(\bar{n}+1)$ and the revival time $t_{\mathrm{rev}} = 4\pi ma^2/\hbar = 2(\bar{n}+1)T_{\mathrm{cl}}$. There is no super-revival time here. We refer to [14] for a systematic study of the temporal behavior of the probability density, $|\langle x \mid J,\gamma\rangle|^2$ with $\gamma = \omega t$, of the square modulus of the autocorrelation, $|\langle J,0 \mid J,\gamma\rangle|^2$, of the mean position and mean "momentum" and their respective dispersions, and so on.

7. Coherent states for continuum dynamics

We now turn to the continuum case. Suppose the quantum system has a Hamiltonian with positive non degenerate purely continuous spectrum,

$$\mathscr{H}\,|\,\mathscr{E}\,\rangle = \mathscr{E}\,|\,\mathscr{E}\,\rangle, \quad 0 \le \mathscr{E} < \bar{\mathscr{E}}, \tag{7.1}$$

where $|\,\mathscr{E}\,\rangle$ denotes some eigendistribution. Mimicking the discrete case, we define the **normalized** coherent states as the continuous superposition

$$|\,J,\gamma\rangle_\rho = \frac{1}{\sqrt{\mathscr{N}_\rho(J)}} \int_0^{\bar{\mathscr{E}}} \frac{J^{\mathscr{E}/2}}{\sqrt{\rho(\mathscr{E})}} e^{-i\gamma\mathscr{E}}\,|\,\mathscr{E}\,\rangle\, d\mathscr{E}. \tag{7.2}$$

The normalization condition reads

$$\mathscr{N}_\rho(J) = \int_0^{\bar{\mathscr{E}}} \frac{J^{\mathscr{E}}}{\rho(\mathscr{E})}\, d\mathscr{E}, \tag{7.3}$$

which has to be finite for, say, $0 \le J < R$. Equation (7.3) gives the ratio $J^{\mathscr{E}}/(\mathscr{N}_\rho(J)\rho(\mathscr{E}))$ a weighting distribution meaning.

These coherent states clearly have Properties P_0 (continuity) and P_1 (temporal stability). For having the resolution of the unity, we first suppose the existence of a nonnegative density $w_\rho(J)$ such that

$$\int_0^R J^{\mathscr{E}} w_\rho(J)\, dJ = \rho(\mathscr{E}). \tag{7.4}$$

Then we have,

$$\mathrm{I} = \int |\,J,\gamma\rangle_{\rho\,\rho}\langle J,\gamma\,|\, d\mu_\rho(J,\gamma) \equiv \int_{-\infty}^{\infty} d\gamma \int_0^R \mathscr{N}_\rho(J) w_\rho(J)\, dJ\, |\,J,\gamma\rangle_{\rho\,\rho}\langle J,\gamma\,|\,. \tag{7.5}$$

Note the simplification in the γ integration afforded by the continuous spectrum. Like in the discrete case, there is an interplay between the functional dependence of the lower symbol and the weighting distribution:

$$\check{\mathscr{H}}(J) = {}_\rho\langle J,\gamma\,|\,\mathscr{H}\,|\,J,\gamma\rangle_\rho = \frac{1}{\mathscr{N}_\rho(J)} \int_0^{\bar{\mathscr{E}}} \frac{\mathscr{E} J^{\mathscr{E}}}{\rho(\mathscr{E})}\, d\mathscr{E} = J\frac{\partial}{\partial J}\log \mathscr{N}_\rho(J) \tag{7.6}$$

$$\text{i.e.,}\quad \mathscr{N}_\rho(J) = \mathscr{N}_\rho(J_0)\exp\Big(\int_{J_0}^{J} \check{\mathscr{H}}(x)\,\frac{dx}{x}\Big). \tag{7.7}$$

8. Coherent states for discrete *and* continuum dynamics

Finally, we briefly present the more general case of an Hamiltonian having both discrete and continuous spectra:

$$\mathscr{H} = \mathscr{H}_D \oplus \mathscr{H}_C, \quad 0 \leq \mathscr{H}_D \leq \Omega,\ \Omega < \mathscr{H}_C : \tag{8.1}$$

We introduce for this situation the following superposition of respective coherent states:

$$\mid J,\gamma;K,\delta;\eta,\phi\rangle_\rho = f(K,\delta-\eta)\mid J,\gamma\rangle_\rho + e^{-i\phi}g(J)\mid K,\delta\rangle_\rho, \tag{8.2}$$

where $\mid J,\gamma\rangle_\rho$ (resp. $\mid K,\delta\rangle_\rho$) takes place for the discrete (resp. continuous) spectrum. Continuity and temporal stability follow from this expression:

$$e^{-i\mathscr{H}t}\mid J,\gamma;K,\delta;\eta,\phi\rangle_\rho = \mid J,\gamma+\omega t;K,\delta+\omega t;\eta+\omega t,\phi+\Omega t\rangle_\rho. \tag{8.3}$$

Establishing the resolution of the unity,

$$\mathrm{I} = \int \mid J,\gamma;K,\delta;\eta,\phi\rangle_{\rho\rho}\langle J,\gamma;K,\delta;\eta,\phi\mid d\lambda, \tag{8.4}$$

$$d\lambda\,(J,\gamma;K,\delta;\eta,\phi) \equiv d\mu_D(J,\gamma)\,d\mu_C(K,\delta)\,h(\eta)\,d\eta\,d\phi/2\pi, \tag{8.5}$$

is more elaborate and explains the presence in (8.2) of the functions f, g, and the parameters η and ϕ. The ranges of values assumed by the latter are respectively $(-\infty,\infty)$ and $[0,2\pi)$. The conditions the three functions f, g, h, have to satisfy are the following:

$$\int |f(K,\delta-\eta)|^2\mid J,\gamma\rangle_{\rho\rho}\langle J,\gamma\mid d\lambda = \mathrm{I}_D, \tag{8.6}$$

$$\int |g(J)|^2\mid K,\delta\rangle_{\rho\rho}\langle K,\delta\mid d\lambda = \mathrm{I}_C, \tag{8.7}$$

$$\int_0^{2\pi} d\phi e^{i\phi}\int g(J)^\star f(K,\delta-\eta)\mid J,\gamma\rangle_{\rho\rho}\langle K,\delta\mid d\lambda = 0, \tag{8.8}$$

$$\text{with} \int_{-\infty}^{\infty} h(\eta)\,d\eta = 1,\ \int |f(K,\delta)|^2\,d\mu_C = 1,\ \int |g(J)|^2\,d\mu_D = 1. \tag{8.9}$$

Again here, a predefinite functional form assigned to the lower symbol $\check{\mathscr{H}}(J,K,\delta,\eta,\phi) \equiv |f(K,\delta-\eta)|^2\check{\mathscr{H}}_D + |g(J)|^2\check{\mathscr{H}}_C$ implies determining constraints for linear combination coefficients and normalization factors, i.e., for the weighting distributions present in the construction.

9. Concluding general expression

Through the successive generalizations of coherent states presented in this paper, we have been becoming gradually aware of the existence of some unifying formalism which underlies all various constructions. So let us conclude by proposing the following general form. Let a quantum system be with one degree of freedom and with Hamiltonian $\mathscr{H}$. We suppose that the spectrum of $\mathscr{H}$, $\mathrm{Sp}(\mathscr{H})$, is bounded from below (as it should be!), and we choose its lower bound equal to zero. Let σ be a **complex** measure with support contained in $\mathrm{Sp}(\mathscr{H})$. We then consider the linear superposition of Hamiltonian eigenstates or eigendistributions $|\,\lambda\,\rangle$, $\mathscr{H}\,|\,\lambda\,\rangle = \lambda\,|\,\lambda\,\rangle$, in a possibly abusive notation:

$$\int |\,\lambda\,\rangle\, d\sigma(\lambda) \equiv |\,\sigma\,\rangle, \tag{9.1}$$

with the following requirements on σ.

Temporal stability:
The complex nature of the measure σ should assure the temporal stability:

$$e^{-i\mathscr{H}t}\,|\,\sigma\,\rangle = |\,e^{-i\lambda t}\sigma\,\rangle. \tag{9.2}$$

Normalization:

$$1 = \langle\,\sigma\,|\,\sigma\,\rangle = \int \Big|\frac{d\sigma}{d\lambda}\Big|^2 d\lambda, \tag{9.3}$$

with again a possibly abusive notation for the "density" $d\sigma/d\lambda$ of the measure σ.

Operator resolution:

$$A = \int |\,\sigma\rangle\langle\sigma\,|\, d\mu(\sigma), \tag{9.4}$$

where A is some positive bounded operator, *e.g.* $A = P\Delta P$ where P is a projector and Δ defines a frame.

Lower symbol identity:

$$\langle\,\sigma\,|\,\mathscr{H}\,|\,\sigma\,\rangle = \check{\mathscr{H}}(\sigma). \tag{9.5}$$

The extension of the formalism presented in this paper to systems with several degrees of freedom is possible, and rather straightforward for completely integrable systems.

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Tensor products of Frobenius manifolds and moduli spaces of higher spin curves

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Abstract. We review progress on the generalized Witten conjecture and some of its major ingredients. This conjecture states that certain intersection numbers on the moduli space of higher spin curves assemble into the logarithm of the τ function of a semiclassical limit of the r-th Gelfand-Dickey (or KdV_r) hierarchy. Additionally, we prove that tensor products of the Frobenius manifolds associated to such hierarchies admit a geometric interpretation in terms of moduli spaces of higher spin structures. We also elaborate upon the analogy with Gromov-Witten invariants of a smooth, projective variety.

Keywords: cohomological field theory, Gromov-Witten invariants, moduli spaces, integrable hierarchies, KdV, Frobenius manifolds

Mathematics Subject Classifications (2000): 14H10,32G15,81T40

In memory of Moshé Flato

0. Introduction

In recent years, there has been a great deal of interaction between mathematics and quantum field theory. One such area has been in the context of topological gravity coupled to topological matter.

The notion of a cohomological field theory (CohFT), due to Kontsevich and Manin [22] is an axiomatization of the expected factorization properties of the correlators appearing in such a theory, cf. [26, 27, 28, 3]. The Gromov-Witten invariants of a smooth, projective variety V provide a rigorous construction of

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a CohFT, one for each V, with a state space $H^\bullet(V)$ and a (Poincaré) metric η. (All cohomology groups in this paper will be assumed to be over $\mathbb{C}$.) The genus zero correlators endow $H^\bullet(V)$ with the structure of a (formal) Frobenius manifold which is a deformation of the cup product, and which yields the structure of quantum cohomology. Furthermore, for two smooth, projective varieties V and V', the quantum cohomology of $V \times V'$ is related to $H^\bullet(V) \otimes H^\bullet(V')$ through a deformation of the Künneth formula. The Gromov-Witten invariants are known to be symplectic invariants of V and are therefore of great mathematical interest.

When V is a point, Witten conjectured [28] and Kontsevich proved [21] that intersection numbers of tautological cohomological classes on the moduli space of stable curves assemble into a τ function of the KdV hierarchy. However, the KdV hierarchy is only the first of a series of integrable hierarchies, one for each integer $r \geq 2$, called the KdV$_r$ or the r-th Gelfand-Dickey hierarchy, where the usual KdV hierarchy is KdV$_2$. Witten stated a generalization [26] of this conjecture where he suggested a construction of moduli spaces and cohomology classes on them whose intersection numbers assemble into a τ function of the KdV$_r$ hierarchy. At the time that his conjecture was formulated, however, the relevant compact moduli spaces had not yet been constructed.

In [14, 15], the moduli space of stable r-spin curves of genus g, $\overline{\mathscr{M}}^{1/r}_{g,n}$, was constructed. In [16], it was proved that certain intersection numbers on $\overline{\mathscr{M}}^{1/r}_{0,n}$ assemble into the genus zero part of the τ function of the KdV$_r$ hierarchy. In addition, there is a (genus zero part of a) CohFT associated to KdV$_r$ whose associated Frobenius manifold has a potential that is polynomial in its flat coordinates. The relevant potential function in genus zero satisfies the string equation, dilaton equation, and topological recursion relations. These proofs are purely algebro-geometric. Axioms have also been formulated for a *virtual class* in higher genus. The existence of a class meeting these axioms gives a CohFT in all genera. For the case of $g = 0$ and all $r > 1$ as well as for the case of $r = 2$ and all $g \geq 0$, this class was constructed in [16], but an algebro-geometric construction of this class for all r and g has yet to be completed.

From a physical perspective, it should not be surprising that the theory of Gromov-Witten invariants and the KdV$_r$ theory should share many features (see the table on next page). Gromov-Witten invariants correspond to a theory of topological gravity with a matter sector arising from the topological sigma model with target variety V. However, one may also consider other choices of matter sector. Witten was led to his KdV$_r$ conjecture by studying the theory of topological gravity coupled to a matter sector arising from a coset model. Even mirror symmetry has an analog in this theory; namely, the Frobenius manifold given by the CohFT associated to KdV$_r$ is isomorphic to that arising from the A_{r-1} singularity, cf. [4, 5, 24, 25].

In this paper, we elaborate upon the analogy between the theory of Gromov-Witten invariants and the KdV$_r$ theory (see the table) using the results in [16].

We construct a geometric "A-model" realization (extending the results in [16]) of the tensor product of the Frobenius manifolds associated to KdV_r in terms of the moduli space of multiple spin structures on stable curves.

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1. Comparison of CohFTs related to stable maps and higher spin curves

This table illustrates some of the similarities and differences between stable maps and higher spin curves. Details and notation will be explained in subsequent sections.

	Stable maps	**Higher spin curves**
Key moduli spaces	Stable maps: $\overline{\mathscr{M}}_{g,n}(V)$	r-spin curves: $\overline{\mathscr{M}}^{1/r}_{g,n}$
State space	$H^\bullet(V)$	$\mathscr{H}^{(r)}$ with basis $\{e_0,\dots,e_{r-2}\}$
Metric	$\eta(\gamma',\gamma'')=\int_V\gamma'\cup\gamma''$	$\eta(e_{m'},e_{m''})=\delta_{m'+m'',r-2}$
Flat identity	1	e_0
Grading	Integral	Fractional
Fundamental class	Virtual fundamental class $[\overline{\mathscr{M}}_{g,n}(V)]^{\mathrm{virt}}$	Orbifold fundamental class $[\overline{\mathscr{M}}^{1/r}_{g,n}]$
"Gromov-Witten" classes	True Gromov-Witten classes: $\mathrm{ev}^*(\boldsymbol{\gamma}):=\mathrm{ev}_1^*\gamma_1\cup\ldots\cup\mathrm{ev}_n^*\gamma_n$	Virtual class $c^{1/r}(\mathbf{m})$: constructed for $r=2$ and all g as well as for $g=0$ and all r.
Gravitational correlators	$\left\langle\tau_{a_1}(\gamma_1)\ldots\tau_{a_n}(\gamma_n)\right\rangle_g$ $=\int_{[\overline{\mathscr{M}}_{g,n}(V)]^{\mathrm{virt}}}\mathrm{ev}^*(\boldsymbol{\gamma})\prod_{i=1}^n\psi_i^{a_i}$	$\left\langle\tau_{a_1}(e_{m_1})\ldots\tau_{a_n}(e_{m_n})\right\rangle_g$ $=\int_{[\overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}]}c^{1/r}\prod_{i=1}^n\psi_i^{a_i}$
Free Energy	Large phase space potential: $\langle\exp(\mathbf{t}\cdot\boldsymbol{\tau})\rangle$	Logarithm of τ function: $\langle\exp(\mathbf{t}\cdot\boldsymbol{\tau})\rangle$
Virasoro algebra action	The Virasoro Conjecture: proved for V a point. Proved for $g=0$ and in some cases for $g=1$.	W_r-algebra conjecture: proved for $g=0$ as well as for $r=2$
Integrable system	KdV_2 when V is a point. Known in $g=0$ and in some cases in $g=1$.	KdV_r hierarchy
(Generalized) Witten conjectures	Proved for V a point.	Proved for $g=0$ as well as for $r=2$.
Cohomological field theory	$\Lambda_{g,n}=\mathrm{st}_*(\mathrm{ev}^*(\boldsymbol{\gamma})\cap[\overline{\mathscr{M}}_{g,n}]^{\mathrm{virt}})$	$\Lambda_{g,n}=p_*c^{1/r}(\mathbf{m})$
WDVV equation	Holds	Holds
String equation	Holds	Holds
TRR	Holds	Holds
Tensor product of CohFTs	Realized by the product $V_1\times V_2$ (Quantum Künneth theorem in $g=0$)	Realized by the fibered product $\overline{\mathscr{M}}^{1/r,1/s}_{g,n}$
Isomorphisms	Mirror symmetry	Versal deformation of A_{r-1} singularity

Remarks 1.1

1. From a geometric point of view the role of the virtual fundamental class of stable maps is played more by the virtual class $c^{1/r}$ than by the orbifold fundamental class. Indeed, most of the real geometric subtlety of the r-spin theory is contained in the construction of $c^{1/r}$, whereas the smoothness of the moduli spaces $\overline{\mathscr{M}}^{1/r}_{g,n}$ implies that the orbifold class is a simple modification of the standard fundamental class.

2. For general V, the generalized Witten conjecture is not precisely formulated, although it is expected that for each V the exponential of the free energy is a τ function for some integrable system. In $g=0$ and for some cases in $g=1$ this integrable system is described in [6, 7].

3. The W_r-algebra contains the Virasoro algebra, and so the corresponding conjecture is a generalization of the Virasoro Conjecture.

2. The KdV$_r$ hierarchy and its semiclassical limit

2.1. The KdV$_r$ hierarchy

In this section we introduce the generalized KdV hierarchies (also called the Gelfand-Dickey hierarchies) and some of their solutions which will be used in the subsequent sections.

Fix an integer $r \geq 2$ and consider differential operators of order r in $D = \frac{i}{\sqrt{r}} \frac{\partial}{\partial x}$ (where the factor $\frac{i}{\sqrt{r}}$ is added for convenience). A monic differential operator of order r, after conjugation by the operator of multiplication by an appropriate function, can be written in the form

$$L = D^r - \sum_{m=0}^{r-2} u_m(x) D^m, \tag{1}$$

where u_m are formal functional variables. Denote by $\mathscr{D}$ the affine space of the operators of type (1).

For every operator $L \in \mathscr{D}$ there exists a unique pseudo-differential operator

$$L^{1/r} = D + \sum_{m>0} w_m D^{-m},$$

such that $(L^{1/r})^r = L$. All coefficients w_m of $L^{1/r}$ are differential polynomials in $u_0, u_1, \ldots, u_{r-2}$.

A pseudodifferential operator $Q = \sum_{m \geq -n} v_m D^{-m}$ can be decomposed as $Q = Q_+ + Q_-$, where $Q_+ = \sum_{m=-n}^{0} v_m D^{-m}$ is the differential part of Q.

For every $k \geq 0$, the operators $Q = (L^{1/r})^k$ and $L = (L^{1/r})^r$ commute yielding $[Q_+, L] = -[Q_-, L]$ and, therefore, $[Q_+, L]$ is a differential operator of order not exceeding $r-2$. Thus the expression $\frac{\partial L}{\partial t} = [Q_+, L]$ gives a meaningful system of differential equations for unknown functions $u_0, \dots, u_{r-2}$. It can be shown that for different values of k these equations are consistent (i.e., the corresponding flows on $\mathscr{D}$ commute) which allows one to define the KdV$_r$ (or the r-th Gelfand-Dickey) hierarchy as the following infinite family of differential equations on $\mathscr{D}$:

$$i\frac{\partial L}{\partial t_n^m} = \frac{k_{n,m}}{\sqrt{r}}\left[(L^{n+\frac{m+1}{r}})_+, L\right], \tag{2}$$

where the constants

$$k_{n,m} = \frac{(-1)^n r^{n+1}}{(m+1)(r+m+1)\dots(nr+m+1)}$$

have been inserted for convenience.

In terms of the unknown functions u_k the equations (2) take the form

$$\frac{\partial u_k}{\partial t_n^m} = S_{k,n}^m,$$

where $S_{k,n}^m$ are polynomials in u_j and their derivatives with respect to x.

There are several simple special cases.

1. When $m+1$ is a multiple of r, the commutator in (2) vanishes and the corresponding equation just means that the functions u_k do not depend on t_n^m.
2. In the case $m = n = 0$ the corresponding equation is $\frac{\partial u_k}{\partial t_0^0} = \frac{\partial u_k}{\partial x}$ and therefore we can identify t_0^0 with x.
3. Finally, when $r = 2$ the KdV$_r$ hierarchy becomes the ordinary KdV hierarchy.

2.2. Potential

In the notation of the previous subsection, we introduce the functions

$$v_n = -\frac{r}{n+1}\operatorname{res}(L^{1/r})^{n+1}, \tag{3}$$

where the residue of a pseudodifferential operator is defined as the coefficient of D^{-1}. The functions v_k can be expressed in terms of u_j by a triangular system of differential polynomials. This means that u_j can be expressed in terms of v_n in a similar way, and we can consider $v_0, v_1, \dots, v_{r-2}$ as a new system of coordinates in $\mathscr{D}$.

We call a function $\widetilde{\Phi}(\mathbf{t})$ in formal variables t_n^m, $n, m \geq 0$, a *potential* of the KdV$_r$ hierarchy if, first, $\widetilde{\Phi}(\mathbf{0}) = 0$ and second, the functions

$$v_m(\mathbf{t}) = \frac{\partial\widetilde{\Phi}(\mathbf{t})}{\partial t_0^0 \partial t_0^m}$$

satisfy the equations (2) with $x = t_0^0$ and u_j is related to v_m via (3) and, finally, $\widetilde{\Phi}(\mathbf{t})$ satisfies the so-called *string equation*

$$\frac{\partial \widetilde{\Phi}(\mathbf{t})}{\partial t_0^0} = \frac{1}{2} \sum_{m,n=0}^{r-2} \eta_{mn} t_0^m t_0^n + \sum_{k=0}^{\infty} \sum_{m=0}^{r-2} t_{k+1}^m \frac{\partial \widetilde{\Phi}(\mathbf{t})}{\partial t_k^m}, \tag{4}$$

where $\eta_{mn} = \delta_{m+n,r-2}$.

It can be shown that the potential $\widetilde{\Phi}(\mathbf{t})$ is uniquely determined by these conditions (cf. [28]).

2.3. Semiclassical approximation

The hierarchy KdV_r (2) has a semiclassical (or dispersionless) limit which is defined in terms of the formalism of Section 2.1 as follows. For a differential operator $L \in \mathscr{D}$ given by (1) denote by $\widetilde{L} = p^r - \sum_{m=0}^{r-2} u_m(x) p^m$ the polynomial in a formal variable p obtained by replacing D with p. The commutator $[L, Q]$ of differential operators will be replaced in (2) by the Poisson bracket

$$\{\widetilde{L}, \widetilde{Q}\} = \frac{\partial \widetilde{L}}{\partial p} \frac{\partial \widetilde{Q}}{\partial x} - \frac{\partial \widetilde{Q}}{\partial p} \frac{\partial \widetilde{L}}{\partial x}.$$

The semiclassical limit of the KdV_r *hierarchy*, KdV_r^s is the system of equations

$$\frac{\partial \widetilde{L}}{\partial t_n^m} = \frac{k_{m,n}}{r} \left\{ \widetilde{L^{n+\frac{m+1}{r}}}, \widetilde{L} \right\} \tag{5}$$

in unknown functions $u_0, \ldots, u_{r-2}$.

The corresponding potential function $\widetilde{\Phi}_0(\mathbf{t})$ is defined as the unique function satisfying the string equation (4), the condition $\widetilde{\Phi}_0(\mathbf{0}) = 0$, and such that the functions $u_0, \ldots, u_{r-2}$ given by (3) and

$$v_m(\mathbf{t}) = \frac{\partial \widetilde{\Phi}_0(\mathbf{t})}{\partial t_0^0 \partial t_0^m}$$

satisfy the equations of the hierarchy (5).

3. The generalized Witten conjecture

In [28], Witten conjectured that the potential $\widetilde{\Phi}(\mathbf{t})$ of the KdV hierarchy is equal to a generating function for the intersection numbers of certain tautological classes on the moduli space of stable curves $\overline{\mathscr{M}}_{g,n}$. This conjecture arose from the study of physical realizations of pure topological gravity and provided an unexpected link between integrable systems and the geometry of these moduli spaces. This conjecture was proved by Kontsevich [21].

To be more precise, let $\overline{\mathcal{M}}_{g,n} := \{[\Sigma, x_1, \ldots, x_n]\}$ be the moduli space of stable curves of genus g with n marked points. That is, Σ is a conformal equivalence class of genus g Riemann surfaces, with (at worst) nodal singularities and $x_1, \ldots, x_n$ are marked points, distinct from each other and the nodes. These data are subject to the stability condition that they must have no infinitesimal automorphisms. This moduli space is a $(3g-3+n)$-dimensional, compact, complex orbifold and is a compactification of the moduli space of genus g Riemann surfaces with n marked points.

This space $\overline{\mathcal{M}}_{g,n}$ is equipped with tautological holomorphic line bundles $\mathcal{L}_{(g,n),i} \to \overline{\mathcal{M}}_{g,n}$ where $i = 1, \ldots, n$, such that the fiber of $\mathcal{L}_{(g,n),i}$ at the point $[\Sigma; x_1, \ldots, x_n]$ is the cotangent space $T^*_{x_i}\Sigma$. Define $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$ to be $c_1(\mathcal{L}_{(g,n),i})$ for all $i = 1, \ldots, n$. Let

$$\langle \tau_{a_1} \ldots \tau_{a_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \ldots \psi_n^{a_n}$$

for all nonnegative integers $a_1, \ldots, a_n \geq 0$. Let $\mathbf{t} := (t_0, t_1, \ldots)$ be formal variables and consider the function $\Phi_g(\mathbf{t}) \in \mathbb{C}[[\mathbf{t}]]$ defined by

$$\Phi_g(\mathbf{t}) := \langle \exp(\mathbf{t} \cdot \boldsymbol{\tau}) \rangle_g = \sum_{n \geq 0} \frac{t_{a_1} \cdots t_{a_n}}{n!} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \tag{6}$$

where $\mathbf{t} \cdot \boldsymbol{\tau} = \sum_{a=0}^{\infty} t_a \tau_a$, the expression $\langle \cdots \rangle_g$ is linear over $\mathbb{C}[[\mathbf{t}]]$, and the exponential function is a formal power series. The summation convention for repeated indices has also been used. Let $\Phi(\mathbf{t}) = \sum_{g \geq 0} \Phi_g(\mathbf{t})$.

Theorem 3.1. (Kontsevich [21]) *The function $\Phi(\mathbf{t})$ defined in (6) is equal to the function $\widetilde{\Phi}(\mathbf{t})$ for* KdV_2.

Kontsevich's proof is fascinating, but it does not explain why such a theorem *should* hold. Kontsevich realizes the function $\widetilde{\Phi}(\mathbf{t})$ as a large N limit of an integral over the space of Hermitian $N \times N$ matrices, a so-called Hermitian matrix model. The terms in the resulting perturbative expansion are indexed by ribbon graphs which, in turn, can be related to a cell decomposition (constructed using Strebel differentials) of a closely related moduli space.

A natural question to ask is whether there is a moduli space and tautological classes on it, whose generating function assembles into the potential $\widetilde{\Phi}$ of the KdV_r hierarchy for general r. Witten [26, 27] stated a conjecture for this case as well. He expected the relevant theory to be topological gravity coupled to a topological matter sector corresponding to a class of topological minimal models which he realized in terms of an $\mathfrak{su}(2)_{r-2}/\mathfrak{u}(1)$ coset model. He suggested a construction of the relevant moduli space and tautological classes. However, the rigorous construction of the appropriate compact moduli space had not yet been completed at the time that the conjecture was formulated.

Recently, Jarvis [14, 15] constructed the moduli space, $\overline{\mathscr{M}}^{1/r}_{g,n}$, called *the moduli space of r-spin curves of genus g with n marked points.* This space is the disjoint union

$$\overline{\mathscr{M}}^{1/r}_{g,n} = \sqcup_{\mathbf{m}} \overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}$$

where the union is over n-tuples of nonnegative integers $\mathbf{m} = (m_1, \dots, m_n)$ where $0 \leq m_i \leq r-1$ for all $i = 1, \dots n$. (In fact, the moduli space $\overline{\mathscr{M}}^{1/r}_{g,n}$ is defined for general integral $\mathbf{m}$ but we will only consider those pieces relevant to the generalized Witten conjecture.)

The spaces $\overline{\mathscr{M}}^{1/r}_{g,n}$ are equipped with tautological classes $\psi_i \in H^2(\overline{\mathscr{M}}^{1/r}_{g,n})$ where $i = 1, \dots, n$. Each ψ_i is the Chern class of a tautological line bundle over $\overline{\mathscr{M}}^{1/r}_{g,n}$. Furthermore, Witten [26] sketched an analytic construction of a tautological class $c^{1/r}$ in $H^D(\overline{\mathscr{M}}^{1/r}_{0,n})$ where $D = \frac{1}{r}(2 - r + \sum_{i=1}^n m_i)$. The correlators

$$\left\langle \tau_{a_1}(e_{m_1}) \dots \tau_{a_n}(e_{m_n}) \right\rangle_g := r \int_{\overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}} c^{1/r} \psi_1^{a_1} \dots \psi_n^{a_n}$$

assemble into a generating function Φ, which is conjectured to coincide with the potential $\widetilde{\Phi}$ of KdV$_r$. The integral above is to be understood in the orbifold sense.[1]

In genus zero a rigorous algebro-geometric construction of this class was given in [16] (see also Section 4.3), and the following result holds.

Theorem 3.2. (The Generalized Witten Conjecture in Genus Zero [16]) *Consider the generating function*

$$\Phi_0(\mathbf{t}) := \langle \exp(\mathbf{t} \cdot \boldsymbol{\tau}) \rangle_0 := \sum_{n \geq 3} \frac{t_{a_1}^{m_1} \cdots t_{a_n}^{m_n}}{n!} \left\langle \tau_{a_1}(e_{m_1}) \dots \tau_{a_n}(e_{m_n}) \right\rangle_0$$

in $\mathbb{Q}[[\mathbf{t}]]$ *where* $\mathbf{t}$ *is the set of variables* t_a^m *for* $a \geq 0$ *and* $m = 0, \dots, r-1$, $\mathbf{t} \cdot \boldsymbol{\tau} := \sum_{a,m} \tau_a(e_m) t_a^m$, *and* $a_i \geq 0$ *and* $0 \leq m_i \leq r-1$. *The function* $\Phi_0(\mathbf{t})$ *which is in fact independent of* t_a^{r-1} *is equal to* $\widetilde{\Phi}_0(\mathbf{t})$, *the (semiclassical) potential of the* KdV$_r^s$ *hierarchy.*

The precise algebro-geometric formulation of this conjecture is, as yet, incomplete in higher genera since the class $c^{1/r}$ has not yet been constructed algebro-geometrically

More detailed descriptions of the above objects will be given in subsequent sections.

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[1] The factor of r can be interpreted as coming from the string coupling constant.

4. The moduli space of curves with multiple spin structures

In this section we review some definitions and facts from [16] about spin curves, and then generalize these to define curves with multiple spin structures.

4.1. Definitions

Definition 4.1 *Let* $(X, p_1, \ldots, p_n)$ *be a nodal n-pointed algebraic curve, and let* $\mathscr{K}$ *be a rank-one, torsion-free sheaf on* X. *A* d-th root of $\mathscr{K}$ of type $\mathbf{m} = (m_1, \ldots, m_n)$ *is a pair* $(\mathscr{E}, b)$ *of a rank-one, torsion-free sheaf* $\mathscr{E}$, *and an* $\mathscr{O}_X$*-module homomorphism*

$$b : \mathscr{E}^{\otimes d} \longrightarrow \mathscr{K} \otimes \mathscr{O}_X(-\sum m_i p_i)$$

with the following properties:

- $d \cdot \deg \mathscr{E} = \deg \mathscr{K} - \sum m_i$;
- *b is an isomorphism on the locus of X where* $\mathscr{E}$ *is locally free;*
- *for every point* $p \in X$ *where* $\mathscr{E}$ *is not free, the length of the cokernel of b at p is* $d - 1$.

For any d-th root $(\mathscr{E}, b)$ of type $\mathbf{m}$, and for any $\mathbf{m}'$ congruent to $\mathbf{m} \mod d$, we can construct a unique d-th root $(\mathscr{E}', b')$ of type $\mathbf{m}'$ simply by taking $\mathscr{E}' = \mathscr{E} \otimes \mathscr{O}(1/d \sum (m_i - m_i')p_i)$. Consequently, the moduli of curves with d-th roots of a bundle $\mathscr{K}$ of type $\mathbf{m}$ is canonically isomorphic to the moduli of curves with d-th roots of type $\mathbf{m}'$. Therefore, unless otherwise stated, we will always assume the type $\mathbf{m}$ of a d-th root has the property that $0 \leq m_i < d$ for all i.

Unfortunately, the moduli space of curves with d-th roots of a fixed sheaf $\mathscr{K}$ is not smooth when d is not prime, and so we must consider not just roots of a bundle, but rather coherent nets of roots [14]. This additional structure suffices to make the moduli space of curves with a coherent net of roots smooth.

Definition 4.2 *Let* $\mathscr{K}$ *be a rank-one, torsion-free sheaf on a nodal n-pointed curve* $(X, p_1, \ldots, p_n)$. *A* coherent net of r-th roots of $\mathscr{K}$ of type $\mathbf{m} = (m_1, \ldots, m_n)$ *consists of the following data:*

- *For every divisor d of r, a rank-one torsion-free sheaf* $\mathscr{E}_d$ *on X;*
- *For every pair of divisors* d', d *of r, such that* d' *divides d, an* $\mathscr{O}_X$*-module homomorphism*

$$c_{d,d'} : \mathscr{E}_d^{\otimes d/d'} \longrightarrow \mathscr{E}_{d'}.$$

These data are subject to the following restrictions.

1. *$\mathscr{E}_1 = \mathscr{K}$ and $c_{1,1} = \mathbf{1}$.*

2. *For each divisor d of r and each divisor d' of d, let $\mathbf{m}'' = (m_1'', \dots, m_n'')$ be such that m_i'' is the unique non-negative integer less than d/d', and congruent to m_i $\mathrm{mod}\ d/d'$. Then the homomorphism $c_{d,d'}$ makes $(\mathscr{E}_d, c_{d,d'})$ into a d/d' root of $\mathscr{E}_{d'}$ of type $\mathbf{m}''$.*

3. *The homomorphisms $\{c_{d,d'}\}$ are compatible. That is, the diagram*

$$\begin{tikzcd}
(\mathscr{E}_d^{\otimes d/d'})^{\otimes d'/d''} \arrow[r, "(c_{d,d'})^{\otimes d'/d''}"] \arrow[dr, "c_{d,d''}"'] & \mathscr{E}_{d'}^{\otimes d'/d''} \arrow[d, "c_{d',d''}"] \\
 & \mathscr{E}_{d''}
\end{tikzcd}$$

commutes for every $d''|d'|d|r$.

If r is prime, then a coherent net of r-th roots is simply an r-th root of $\mathscr{K}$. Moreover, if $\mathscr{E}_d$ is locally free of type $\mathbf{m}$, then for $d'|d$, if $\mathbf{m} \equiv \mathbf{m}' \mod d'$ and $0 \leq m_i' < d'$, then the sheaf $\mathscr{E}_{d'}$ is isomorphic to $\mathscr{E}_d^{\otimes d/d'} \otimes \mathscr{O}(\frac{1}{d'} \sum (m_i - m_i') p_i)$ and $c_{d,d'}$ is uniquely determined up to an automorphism of $\mathscr{E}_{d'}$.

Definition 4.3 *An n-pointed r-spin curve of type $\mathbf{m} = (m_1, \dots, m_n)$ is an n-pointed nodal curve $(X, p_1, \dots, p_n)$ with a coherent net of r-th roots of ω_X of type $\mathbf{m}$, where ω_X is the canonical (dualizing) sheaf of X. An r-spin curve is called* smooth, *if X is smooth, and it is called* stable, *if X is stable.*

Example 4.4 Smooth 2-spin curves of type $\mathbf{m} = \mathbf{0}$ correspond to classical spin curves (a curve and a theta-characteristic) *together with an explicit isomorphism* $\mathscr{E}_2^{\otimes 2} \xrightarrow{\sim} \omega$.

Definition 4.5 *An isomorphism of r-spin curves*

$$(X, p_1, \dots, p_n, \{\mathscr{E}_d, c_{d,d'}\}) \xrightarrow{\sim} (X', p_1', \dots, p_n', \{\mathscr{E}_d', c_{d,d'}'\})$$

of the same type $\mathbf{m}$ is an isomorphism of pointed curves

$$\tau : (X, p_1, \dots, p_n) \xrightarrow{\sim} (X', p_1', \dots, p_n')$$

and a family of isomorphisms $\{\beta_d : \tau^* \mathscr{E}_d' \xrightarrow{\sim} \mathscr{E}_d\}$, with β_1 being the canonical isomorphism $\tau^* \omega_{X'}(-\sum_i m_i p'_i) \xrightarrow{\sim} \omega_X(-\sum m_i p_i)$, and such that the β_d are compatible with all the maps $c_{d,d'}$ and $\tau^* c_{d,d'}'$.

Every r-spin structure on a smooth curve X is determined up to isomorphism by a choice of a line bundle $\mathscr{E}_r$, such that $\mathscr{E}_r^{\otimes r} \cong \omega_X(-\sum m_i p_i)$. In particular, if X has no automorphisms, then the set of isomorphism classes of r-spin structures of a fixed type $\mathbf{m}$ is a principal $\mathrm{Jac}\,_r X$-bundle, where $\mathrm{Jac}\,_r X$ is the group of r-torsion points of the Jacobian of X, thus there are r^{2g} such isomorphism classes.

Example 4.6 If $g = 1$ and $\mathbf{m} = \mathbf{0}$, then $\omega_X \cong \mathscr{O}_X$ and a smooth r-spin curve is just an r-torsion point of the elliptic curve X, again with an explicit isomorphism of $\mathscr{E}_r^{\otimes r} \xrightarrow{\sim} \mathscr{O}_X$. In particular, the stack of stable r-spin curves of genus one and type $\mathbf{0}$ forms a gerbe over the disjoint union of modular curves $\coprod_{d|r} X_1(d)$.

Definition 4.7 *The moduli space of stable n-pointed, r-spin curves of genus g and type* $\mathbf{m}$ *is denoted* $\overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}}$. *The disjoint union* $\coprod_{\substack{\mathbf{m} \\ 0 \le m_i < r}} \overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}}$ *is denoted* $\overline{\mathscr{M}}_{g,n}^{1/r}$.

Remark 4.8 As mentioned above, no geometric information is lost by restricting $\mathbf{m}$ to the range $0 \le m_i \le r-1$, since when $\mathbf{m} \equiv \mathbf{m}' \mod r$ every r-spin curve of type $\mathbf{m}$ naturally gives an r-spin curve of type $\mathbf{m}'$ simply by

$$\mathscr{E}_d \mapsto \mathscr{E}_d \otimes \mathscr{O}(\sum \frac{m_i - m'_i}{d} p_i),$$

and thus $\overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}}$ is canonically isomorphic to $\overline{\mathscr{M}}_{g,n}^{1/r,\mathbf{m}'}$.

More generally, if $\mathbf{r} = (r^1, \ldots, r^k)$ is a k-tuple of integers $r^i \ge 2$, and if $\bar{\mathbf{m}} := (\mathbf{m}^1, \ldots, \mathbf{m}^k)$, $\mathbf{m}^j := (m_1^j, \ldots, m_n^j)$, for nonnegative integers m_i^j we define $\mathbf{r}$-spin curves of type $\bar{\mathbf{m}}$ as follows:

Definition 4.9 *An n-pointed* $\mathbf{r}$*-spin curve of type* $\bar{\mathbf{m}}$ *is an n-pointed, nodal curve* $(X, p_1, \ldots, p_n)$ *with a k-tuple of* $\mathbf{r}$*-spin structures of type* $\bar{\mathbf{m}}$*, that is, a k-tuple whose i-th term is a coherent net* $\mathscr{N}_i := \{\mathscr{E}_{d^i}, c_{d^i,d^{i'}}\}$ *of* r^i*-th roots of* ω_X *of type* $\mathbf{m}^i = (m_1^i, \ldots, m_n^i)$. *Here, of course, the* d^i *run over all positive integers dividing* r^i, *and the* $d^{i'}$ *run over all positive divisors of* d^i.

Definition 4.10 *We denote by* $\overline{\mathscr{M}}_{g,n}^{1/\mathbf{r},\bar{\mathbf{m}}}$ *the moduli space of genus g, n-pointed, stable* $\mathbf{r}$*-spin curves of type* $\bar{\mathbf{m}}$ *and we denote by* $\overline{\mathscr{M}}_{g,n}^{1/\mathbf{r}}$ *the disjoint union*

$$\overline{\mathscr{M}}_{g,n}^{1/\mathbf{r}} := \coprod_{\substack{\bar{\mathbf{m}} = (\mathbf{m}^1, \ldots, \mathbf{m}^k) \\ \mathbf{m}^i = (m_1^i, \ldots, m_n^i) \\ 0 \le m_j^i < r^i}} \overline{\mathscr{M}}_{g,n}^{1/\mathbf{r},\bar{\mathbf{m}}}.$$

4.2. Basic properties of the moduli space $\overline{\mathscr{M}}^{1/\mathbf{r},\bar{\mathbf{m}}}_{g,n}$

These spaces are endowed with a number of canonical morphisms, including the projections

$$q_i : \overline{\mathscr{M}}^{1/\mathbf{r},\bar{\mathbf{m}}}_{g,n} \longrightarrow \overline{\mathscr{M}}^{1/r^i,\mathbf{m}^i}_{g,n}$$

and

$$p_i : \overline{\mathscr{M}}^{1/r^i,\mathbf{m}^1}_{g,n} \longrightarrow \overline{\mathscr{M}}_{g,n}.$$

It is clear that

$$\overline{\mathscr{M}}^{1/\mathbf{r},\bar{\mathbf{m}}}_{g,n} = \prod_{i=1}^{k} \overline{\mathscr{M}}^{1/r^i,\mathbf{m}^i}_{g,n} \tag{7}$$

where $\prod_i$ denotes the fibered product over $\overline{\mathscr{M}}_{g,n}$ with respect to the maps p_i.

In [14] it is shown that $\overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n}$ is a smooth Deligne-Mumford stack (orbifold), finite over $\overline{\mathscr{M}}_{g,n}$, with a projective coarse moduli space. Consequently, $\overline{\mathscr{M}}^{1/\mathbf{r},\bar{\mathbf{m}}}_{g,n}$ has the same properties.

There is one other canonical morphism associated to these spaces; namely, when d divides r, the morphism

$$[r/d] : \overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n} \longrightarrow \overline{\mathscr{M}}^{1/d,\mathbf{m}''}_{g,n} \text{ and } [r/d] : \overline{\mathscr{M}}^{1/r}_{g,n} \longrightarrow \overline{\mathscr{M}}^{1/d}_{g,n},$$

which forgets all of the roots and homomorphisms in the net of r-th roots except those associated to divisors of d. Here, as above, $\mathbf{m}''$ is congruent to $\mathbf{m} \mod d$ and has $0 \leq m''_i < d$. In the case that the underlying curve is smooth, this is equivalent to replacing the line bundle $\mathscr{E}_r$ by its r/d-th tensor power (and tensoring with $\mathscr{O}(1/d \sum (m_i - m''_i) p_i)$).

Remark 4.11 It is important to keep in mind that the space $\overline{\mathscr{M}}^{1/rs}_{g,n}$ is not isomorphic to $\overline{\mathscr{M}}^{(1/r,1/s)}_{g,n} = \overline{\mathscr{M}}^{1/r}_{g,n} \times_{\overline{\mathscr{M}}_{g,n}} \overline{\mathscr{M}}^{1/s}_{g,n}$, even when r and s are relatively prime. However, on the open locus of smooth spin curves we do have $\mathscr{M}^{1/rs}_{g,n} \cong \mathscr{M}^{(1/r,1/s)}_{g,n}$, whenever r and s are relatively prime [14].

Throughout this paper we will denote the universal curve by $\pi : \mathscr{C}^{1/\mathbf{r}}_{g,n} \longrightarrow \overline{\mathscr{M}}^{1/\mathbf{r}}_{g,n}$. As in the case of the moduli space of stable curves, the universal curve possesses canonical sections $\sigma_i : \overline{\mathscr{M}}^{1/\mathbf{r}}_{g,n} \longrightarrow \mathscr{C}^{1/\mathbf{r}}_{g,n}$ for $i = 1, \ldots, n$.

4.3. Cohomology classes

As in the case of stable maps and stable curves we have some tautological cohomology classes. Most important for our purposes are the classes

$$\psi_i := c_1(\mathscr{L}_{(g,n),i}) = c_1(\sigma_i^*(\omega_\pi)),$$

the first Chern class of the restriction of the canonical (relative dualizing) sheaf ω_π to the image of the i-th section. Of course, ψ_i on $\overline{\mathscr{M}}^{1/\mathbf{r},\bar{\mathbf{m}}}_{g,n}$ is just the pullback $p^* \psi_i$

of the usual class ψ_i on $\overline{\mathscr{M}}_{g,n}$ because $\mathscr{L}_{(g,n),i} = \sigma_i^*(\omega_\pi) = p^*\sigma_i^*(\omega_{(g,n)})$, where $\omega_{(g,n)}$ is the relative dualizing sheaf of the universal curve over $\overline{\mathscr{M}}_{g,n}$.

In addition to the tautological classes, we will also need a class $c^{1/r}$ which plays the role of Gromov-Witten classes $(ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n)) \cap ([\overline{\mathscr{M}}_{g,n}(V,\beta)]^{virt}))$ in the case of stable maps.

The class $c^{1/r}$ will be called the *virtual class*, and in the case of $g=0$ it has a simple description as the Euler class of the pushforward of the r-th root bundle from the universal curve to $\overline{\mathscr{M}}^{1/r}_{g,n}$.

Definition 4.12 *If $g=0$, let $\mathscr{E}^i_{r^i}$ be the r^i-th root of ω_π associated to the i-th net $\mathscr{N}_i$ of the universal* **r***-spin structure $(\mathscr{N}_1,\dots,\mathscr{N}_k)$ on the universal curve $\pi : \mathscr{C}^{1/\mathbf{r}}_{0,n} \longrightarrow \overline{\mathscr{M}}^{1/\mathbf{r}}_{0,n}$. Let $c^{1/\mathbf{r}}_{0,n}$ be the Euler class (top Chern class) of the pushforward of $\mathscr{E}^1_{r^1} \oplus \cdots \oplus \mathscr{E}^k_{r^k}$*

$$c^{1/\mathbf{r}}_{0,n} = e(-R\pi_*(\mathscr{E}^1_{r^1} \oplus \cdots \oplus \mathscr{E}^k_{r^k})).$$

Here $R\pi_*\mathscr{F}$ is the K-theoretic pushforward $\pi_*\mathscr{F} - R^1\pi_*\mathscr{F}$. This class is well defined because in genus zero $\mathscr{E}^i_{r^i}$ is *convex* [16], that is $\pi_*\mathscr{E}^i_{r^i} = 0$, and so $R^1\pi_*\mathscr{E}^i_{r^i}$ is a vector bundle of rank $D_i = \chi(\mathscr{E}^i_{r^i}) = ((2-r^i) + \sum_j m^i_j)/r^i$. Thus $R\pi_*(\mathscr{E}^1_{r^1} \oplus \cdots \oplus \mathscr{E}^k_{r^k})$ is also a vector bundle and has a top Chern class.

Let us consider the case where $k=1$ and $r \geq 2$. The class $c^{1/r}$ on $\overline{\mathscr{M}}^{1/r}_{0,n}$ defined above is precisely the class described in Theorem 3.2 (the generalized Witten conjecture).

Lemma 4.13 *When the moduli space $\overline{\mathscr{M}}^{1/\mathbf{r}}_{g,n}$ of curves with multiple spin structures is viewed as a fibered product, as in (7), then the virtual class $c^{1/\mathbf{r}}$ is the product of the pullbacks of the virtual classes from each of the factors:*

$$c^{1/\mathbf{r}} = q_1^*c^{1/r^1} \cup q_2^*c^{1/r^2} \cup \cdots \cup q_n^*c^{1/r^n}.$$

Proof. Since the maps $p_i : \overline{\mathscr{M}}^{1/r^i}_{g,n} \longrightarrow \overline{\mathscr{M}}_{g,n}$ are flat, since $\overline{\mathscr{M}}^{1/\mathbf{r}}_{g,n}$ is a fibered product, and since $\mathscr{E}^i_{r^i}$ is the pullback of the universal r^i-th root on $\mathscr{C}^{1/r^i}_{0,n}$ via q_i, we have

$$\begin{aligned} c^{1/\mathbf{r}} &= e(-(R\pi_*(\mathscr{E}^1_{r^1} \oplus \cdots \oplus \mathscr{E}^k_{r^k}))) \\ &= e(-(q_1^*R\pi_*\mathscr{E}^1_{r^1} \oplus \cdots \oplus q_k^*R\pi_*\mathscr{E}^k_{r^k})) \\ &= q_1^*c^{1/r^1} \cup q_2^*c^{1/r^2} \cup \ldots q_n^*c^{1/r^n}. \end{aligned}$$

■

Remark 4.14 In [16] it is shown that in genus zero the virtual class $c^{1/r}$ has a number of properties resembling those of the Gromov-Witten classes. One additional property that is unique to this construction, and which plays an important role, is the fact that $c^{1/r}$ vanishes on $\overline{\mathscr{M}}_{g,n}^{\mathbf{1/r,m}}$ whenever any one of the m^i_j is equal to $r^i - 1$. Also it vanishes in some other special cases.

5. Cohomological field theories and Frobenius manifolds

5.1. Operads and cohomological field theories

The objects defined in the previous section allow a construction of a $g = 0$ cohomological field theory (CohFT) whose potential function coincides with the potential function of KdV_r and gives rise to an associated $(r-1)$ dimensional Frobenius manifold. It is precisely this property, together with the topological recursion relations (TRR) which was used to prove the generalized Witten conjecture in genus zero. It is also the framework in which the notion of tensor product most readily appears. These constructions fit into the framework of stable graphs and operads which we will now describe.

Consider a connected graph Γ with n tails (or legs) indexed by numbers $1, \dots, n$, such that each vertex v is labeled by a nonnegative integer $g(v)$ (called the *genus of* v). The *genus* $g(\Gamma)$ of Γ is defined as $g(\Gamma) = \sum_{v \in V(\Gamma)} g(v) + b_1(\Gamma)$, where $V(\Gamma)$ is the set of vertices of Γ and $b_1(\Gamma)$ is the first Betti number (number of loops) of Γ. Such a graph Γ is called a *stable graph* of genus g with n tails if for every $v \in V(\Gamma)$ with $g(v) = 0$ its valence (denoted by $n(v)$) satisfies $n(v) \geq 3$ and if $g(v) = 1$ then $n(v) \geq 1$. In particular, genus zero stable graphs are trees whose vertices are all labeled by 0.

Each stable curve $[\Sigma, x_1, \dots, x_n] \in \overline{\mathscr{M}}_{g,n}$ is associated to a stable graph of genus g with n tails in the following way.

To each irreducible component, associate a vertex, and draw as many half-edges emerging from this vertex as there are *special points* (either marked points or nodes) on that component. Label vertex v by an integer $g(v)$ equal to the genus of that irreducible component. Connect any two half edges if they are associated to the same node. Finally, label the remaining half-edges (tails) by an integer from $1, \dots, n$ corresponding to the associated marked point.

Let Γ be a stable graph of genus g with n tails. The stratification of $\overline{\mathscr{M}}_{g,n}$ yields *gluing maps*

$$\rho_\Gamma : \prod_{v \in V(\Gamma)} \overline{\mathscr{M}}_{g(v),n(v)} \to \overline{\mathscr{M}}_{g,n}.$$

This map is obtained by composing the maps $\chi : \prod_{v \in V(\Gamma)} \overline{\mathscr{M}}_{g(v),n(v)} \to \overline{\mathscr{M}}_\Gamma$ and $i : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{g,n}$. The space $\overline{\mathscr{M}}_\Gamma$ is the closure of the locus of points in $\overline{\mathscr{M}}_{g,n}$ whose associated stable graph is Γ. The map χ is the quotient map corresponding to the action of $\mathrm{Aut}(\Gamma)$, the automorphism group of Γ, and i is the inclusion map. The symmetric group $S_{n(v)}$ acts on the space $\overline{\mathscr{M}}_{g(v),n(v)}$ by permuting the marked points. Similarly, S_n acts on $\overline{\mathscr{M}}_{g,n}$. Clearly, the map ρ_Γ is equivariant under these actions.

The collection $\overline{\mathscr{M}} := \{\overline{\mathscr{M}}_{g,n}\}$, together with the symmetric group actions and the maps ρ_Γ, is the model example of a *modular operad* [11]. Restricting to genus zero stable graphs, one obtains a *cyclic operad*, a refinement of the notion of an operad. The modular operad structure on $\overline{\mathscr{M}}$ induces a modular operad structure on the homology $H_\bullet(\overline{\mathscr{M}}) := \{H_\bullet(\overline{\mathscr{M}}_{g,n})\}$. From this perspective, a cohomological field theory (CohFT) is simply a vector space $\mathscr{H}$ with a metric η, which is an algebra over $H_\bullet(\overline{\mathscr{M}})$. The algebra maps $H_\bullet(\overline{\mathscr{M}}_{g,n}) \to T^n\mathscr{H}^*$ (where $T^n\mathscr{H}^*$ denotes the n-fold tensor product of $\mathscr{H}^*$) are the correlators of the theory. However, in the context of algebraic geometry, it turns out to be more useful to use the dual definition of a cohomological field theory, which is given in terms of cohomology.

Definition 5.1 *A cohomological field theory (CohFT) of rank d (denoted by $(\mathscr{H},\eta,\Lambda)$ or just $(\mathscr{H},\eta)$) is a d-dimensional vector space $\mathscr{H}$ with a metric η and a collection of forms $\Lambda := \{\Lambda_{g,n}\}$ where*

$$\Lambda_{g,n} \in H^\bullet(\overline{\mathscr{M}}_{g,n}) \otimes T^n\mathscr{H}^* = Hom(T^n\mathscr{H}, H^\bullet(\overline{\mathscr{M}}_{g,n})) \tag{8}$$

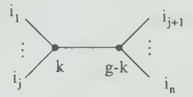
are defined for stable pairs (g,n) and satisfy the following axioms **C1–C3**. *Let $\{e_0,\ldots,e_{d-1}\}$ be a fixed basis of $\mathscr{H}$, and let $\eta^{\mu\nu}$ be the inverse of the matrix of the metric η in this basis. We use the summation convention in the following.*

C1. *The form $\Lambda_{g,n}$ is invariant under the action of the symmetric group S_n.*

C2. *Let*

$$\rho_{\Gamma_{\text{tree}}} : \overline{\mathscr{M}}_{k,j+1} \times \overline{\mathscr{M}}_{g-k,n-j+1} \longrightarrow \overline{\mathscr{M}}_{g,n} \tag{9}$$

be the gluing map corresponding to the stable graph

$\Gamma_{\text{tree}} =$ 

then the forms $\Lambda_{g,n}$ satisfy the composition property

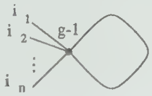
$$\begin{aligned}\rho^*_{\Gamma_{\text{tree}}}\Lambda_{g,n}(\gamma_1,\gamma_2\ldots,\gamma_n) \ &= \\ \Lambda_{k,j+1}(\gamma_{i_1},\ldots,\gamma_{i_j},e_\mu)\eta^{\mu\nu} \ &\otimes \ \Lambda_{g-k,n-j+1}(e_\nu,\gamma_{i_{j+1}},\ldots,\gamma_{i_n})\end{aligned} \tag{10}$$

for all $\gamma_i \in \mathscr{H}$.

C3. *Let*

$$\rho_{\Gamma_{\text{loop}}} : \overline{\mathscr{M}}_{g-1,n+2} \longrightarrow \overline{\mathscr{M}}_{g,n} \tag{11}$$

be the gluing map corresponding to the stable graph

$\Gamma_{\text{loop}} =$  (12) *then*

$$\rho^*_{\Gamma_{\text{loop}}} \Lambda_{g,n}(\gamma_1, \gamma_2, \ldots, \gamma_n) = \Lambda_{g-1,n+2}(\gamma_1, \gamma_2, \ldots, \gamma_n, e_\mu, e_\nu)\, \eta^{\mu\nu}. \tag{13}$$

The pair $(\mathscr{H}, \eta)$ *is called the* state space *of the CohFT. The maps* $H_\bullet(\overline{\mathscr{M}}_{g,n}) \to T^n\mathscr{H}^*$ *given by* $[c] \mapsto \int_{[c]} \Lambda_{g,n}$ *are called the* correlators *of* the CohFT.

An element $e_0 \in \mathscr{H}$ *is called* a flat identity *of the CohFT, if in addition, the following equations hold.*

C4a. *For all* γ_i *in* $\mathscr{H}$ *we have*

$$\Lambda_{g,n+1}(\gamma_1, \ldots, \gamma_n, e_0) = \pi^*\Lambda_{g,n}(\gamma_1, \ldots, \gamma_n), \tag{14}$$

where $\pi : \overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n}$ *is the universal curve on* $\overline{\mathscr{M}}_{g,n}$ *and*

C4b.

$$\int_{\overline{\mathscr{M}}_{0,3}} \Lambda_{0,3}(\gamma_1, \gamma_2, e_0) = \eta(\gamma_1, \gamma_2). \tag{15}$$

A CohFT with flat identity is denoted by $(\mathscr{H}, \eta, \Lambda, e_0)$.

A genus g CohFT *on the state space* $(\mathscr{H}, \eta)$ *is the collection of forms* $\{\Lambda_{\tilde{g},n}\}_{\tilde{g}\leq g}$ *that satisfy only those of the equations (10), (13), (14), and (15), where* $\tilde{g} \leq g$.

Remark 5.2 The state space $\mathscr{H}$ of a CohFT can, in general, be $\mathbb{Z}_2$-graded, but for simplicity we will assume that $\mathscr{H}$ contains only even elements as is the case for KdV_r.

Let Γ be a genus g stable graph with n tails, then, since the map ρ_Γ can be constructed from gluing morphisms (9) and (11), the forms $\Lambda_{g,n}$ satisfy the restriction property

$$\rho^*_\Gamma \Lambda_{g,n} = \eta_\Gamma^{-1}\Big(\bigotimes_{v\in V(\Gamma)} \Lambda_{g(v),n(v)} \Big) \tag{16}$$

where

$$\eta_\Gamma^{-1} : \bigotimes_{v\in V(\Gamma)} T^{n(v)}\,\mathscr{H}^* \to T^n\,\mathscr{H}^*$$

contracts the factors $T^n\mathscr{H}^*$ by means of the inverse of the metric η and successive application of equations (10) and (13) .

Definition 5.3 *The* potential function *of the CohFT* $(\mathscr{H}, \eta, \Lambda)$ *is a formal series* $\Phi \in \mathbb{C}[[\mathscr{H}]]$ *given by*

$$\Phi(\mathbf{x}) := \sum_{g=0}^{\infty} \Phi_g(\mathbf{x}), \tag{17}$$

where

$$\Phi_g(\mathbf{x}) := \sum_n \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \langle \Lambda_{g,n}, \mathbf{x}^{\otimes n} \rangle .$$

Here $\langle \cdots \rangle$ *denotes evaluation, the sum over* n *is understood to be over the stable range, and* $\mathbf{x} = \sum_\alpha x^\alpha e_\alpha$, *where* $\{e_\alpha\}$ *is a basis of* $\mathcal{H}$.[2]

All of the information of a genus zero CohFT is encoded in its potential.

Theorem 5.4. **([22, 24])** *An element* Φ_0 *in* $\mathbb{C}[[\mathcal{H}]]$ *is the potential of a rank* d, *genus zero CohFT* $(\mathcal{H}, \eta)$ *if and only if it contains only terms which are of cubic and higher order in the coordinates* $x^0, \ldots, x^{d-1}$ *(corresponding to a basis* $\{e_0, \ldots e_{d-1}\}$ *of* $\mathcal{H}$*) and it satisfies the* associativity, or WDVV *(Witten-Dijkgraaf-Verlinde*[2]*) equation*

$$\partial_a \partial_b \partial_e \Phi_0 \, \eta^{ef} \, \partial_f \partial_c \partial_d \Phi_0 = \partial_b \partial_c \partial_e \Phi_0 \, \eta^{ef} \, \partial_f \partial_a \partial_d \Phi_0,$$

where η^{ab} *is the inverse of the matrix of* η *in the basis* $\{e_a\}$, ∂_a *is derivative with respect to* x^a, *and the summation convention has been used.*

Conversely, a genus zero CohFT structure on $(\mathcal{H}, \eta)$ *is uniquely determined by its potential* Φ_0, *which must satisfy the WDVV equation.*

A formal power series Φ which satisfies the WDVV equations determines a *formal Frobenius manifold*, so the theorem shows that a genus zero CohFT with flat identity is equivalent to endowing the state space $(\mathcal{H}, \eta)$ with the structure of a formal Frobenius manifold [5, 13, 24]. The theorem follows from the work of Keel [20] who proved that $H^\bullet(\overline{\mathcal{M}}_{0,n})$ is generated by boundary classes, and that all relations between boundary divisors arise from lifting the basic codimension one relation on $\overline{\mathcal{M}}_{0,4}$.

5.2. Gromov-Witten invariants

The best known examples of CohFTs come from the Gromov-Witten classes of a smooth, projective variety V, where the state space $\mathcal{H}$ is $H^\bullet(V)$, and η is the Poincaré pairing. Let $\overline{\mathcal{M}}_{g,n}(V)$ be the moduli space of stable, genus g maps into V with n marked points, a compactification of the moduli space of holomorphic maps $f : \Sigma \to V$ from a genus g Riemann surface Σ with n marked points $(x_1, \ldots, x_n)$ into V. There are canonical evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(V) \to V$ corresponding to evaluating f at x_i. There is also the stabilization morphism $\mathrm{st} : \overline{\mathcal{M}}_{g,n}(V) \to \overline{\mathcal{M}}_{g,n}$ which "forgets" f. Finally, the space $\overline{\mathcal{M}}_{g,n}(V)$ has a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{virt}}$ in the Chow group of $\overline{\mathcal{M}}_{g,n}(V)$. In this situation, $(H^\bullet(V), \eta, \Lambda, \mathbf{1})$ is a CohFT with flat identity, where $\mathbf{1}$ is the usual unit element and

$$\Lambda_{g,n}(\gamma_1, \ldots, \gamma_n) := \mathrm{st}_*(\mathrm{ev}_1^* \gamma_1 \cup \ldots \cup \mathrm{ev}_n^* \gamma_n \cap [\overline{\mathcal{M}}_{g,n}(V)]^{\mathrm{virt}})$$

---

[2] The *string coupling constant* factor of λ^{2g-2} in front of $\Phi_g(\mathbf{x})$ has been suppressed to avoid notational clutter.

for $\gamma_i \in H^\bullet(V)$. In the case of $g = 0$, the correlators endow $H^\bullet(V)$ with a multiplication, which is a deformation of the cup product known as the quantum cohomology of V.

The *large phase space potential* $\Phi(\mathbf{t})$ is the sum of formal power series $\Phi_g(\mathbf{t})$ for $g \geq 0$ where

$$\Phi(\mathbf{t})_g := \langle \exp(\mathbf{t}\cdot\boldsymbol{\tau}) \rangle_g = \sum \frac{t_{a_1}^{m_1}\dots t_{a_n}^{m_n}}{n!} \langle \tau_{a_1}(e_{m_1})\cdots\tau_{a_n}(e_{m_n}) \rangle_g$$

for a given basis $\{e_m\}$ of $H^\bullet(V)$ and

$$\langle \tau_{a_1}(e_{m_1})\cdots\tau_{a_n}(e_{m_n}) \rangle_g = \int_{[\overline{\mathscr{M}}_{g,n}(V)]^{\mathrm{virt}}} \mathrm{ev}_1^* e_{m_1} \psi_1^{a_1} \dots \mathrm{ev}_n^* e_{m_n} \psi_n^{a_n}.$$

Setting t_a^m to zero and $x^m := t_0^m$, one recovers the potential $\Phi(\mathbf{x})$ of the CohFT associated to Λ which is the generating function for the Gromov-Witten invariants of V.[3]

If V is a point then $\overline{\mathscr{M}}_{g,n}(V)$ reduces to $\overline{\mathscr{M}}_{g,n}$ and one has Kontsevich's theorem, which identifies Φ with the potential of the KdV$_2$ hierarchy. For general V a similar relation is expected, but the corresponding integrable system has only been described in $g = 0$ [5], and in some cases, in $g = 1$ [7]. However, there is a related conjecture by Eguchi, Hori, Xiong, and S. Katz [8], which states that the exponential of $\Phi(\mathbf{t})$ satisfies a highest weight condition for an action of the Virasoro algebra. The evidence for this conjecture is growing [9, 10, 12].

5.3. The KdV$_r$ Frobenius manifold

Consider an integer $r \geq 2$. Let $(\mathscr{H}^{(r)}, \eta)$ be an $(r-1)$-dimensional vector space $\mathscr{H}^{(r)}$ with basis $\{e_0, \dots e_{r-2}\}$ and a metric $\eta(e_{m'}, e_{m''}) = \delta_{m'+m'', r-2}$ such that $m', m'' = 0, \dots, r-2$.

Theorem 5.5. **([16])** *For each integer $r \geq 2$, the pair $(\mathscr{H}^{(r)}, \eta)$ is a Frobenius manifold (and thus defines a $g = 0$ CohFT) with flat identity e_0, Euler vector field*

$$E := \sum_{m=0}^{r-2} (\frac{m}{r} - 1) x^m \frac{\partial}{\partial x^m}$$

and

$$\Lambda_{0,n}(e_{m_1}, \dots, e_{m_n}) = r p_* c^{1/r}$$

where $p : \overline{\mathscr{M}}^{1/r,\mathbf{m}}_{g,n} \to \overline{\mathscr{M}}_{g,n}$, $\mathbf{m} := (m_1, \dots, m_n)$, and $0 \leq m_i \leq r-2$. The corresponding potential function $\Phi_0(\mathbf{x})$ (of the Frobenius manifold) is given by the formula

$$\Phi_0(\mathbf{x}) := \sum_{n \geq 3} \frac{1}{n!} x^{m_1} \dots x^{m_n} r \int_{\overline{\mathscr{M}}^{1/r,\mathbf{m}}_{0,n}} c^{1/r},$$

---

[3] To avoid notational clutter, factors of q^β where $\beta \in H_2(V, \mathbb{Z})$ in the Novikov ring have been suppressed.

where $\mathbf{x} := (x^0, \ldots, x^{r-2})$ *are coordinates corresponding to the basis* $\{e_0, \ldots, e_{r-2}\}$ *and* $\mathbf{m} := (m_1, \ldots, m_n)$*. Summation over* $m_i = 0, \ldots, r-2$ *is assumed, and the integral is understood in the sense of orbifolds.*

Furthermore, the potential $\Phi_0(\mathbf{x})$ *is equal to the large phase space potential* $\Phi_0(\mathbf{t})$ *from Theorem 3.2 when* t_a^m *is set to zero for* $a > 0$ *and* $x^m := t_0^m$ *for all* $m = 0 \ldots, r-2$*. The potential* $\Phi_0(\mathbf{t})$ *satisfies the* $g = 0$ *topological recursion relations (TRR)*

$$\frac{\partial^3 \Phi_0}{\partial t^{m_1}_{a_1+1} \partial t^{m_2}_{a_2} t^{m_3}_{a_3}} = \sum_{m_+, m_-} \frac{\partial^2 \Phi_0}{\partial t^{m_1}_{a_1} \partial t^{m_+}_0} \eta^{m_+, m_-} \frac{\partial^3 \Phi_0}{\partial t^{m_-}_0 \partial t^{m_2}_{a_2} \partial t^{m_3}_{a_3}}.$$

The potential also satisfies the string equation, equation (4), where the symbol $\widetilde{\Phi}(\mathbf{t})$ *is replaced by* $\Phi_0(\mathbf{t})$*.*

This result was conjectured by Witten [26, 27] before the proper moduli spaces had even been constructed. The proof of the $g = 0$ CohFT property uses intersection theory on $\overline{\mathscr{M}}^{1/r}_{0,n}$ which is rather involved because of its orbifold (or stacky) nature. The proof of the topological recursion relations follows from the properties of the classes involved when they are restricted to boundary strata and from a presentation of the ψ classes in terms of boundary classes. The string equation follows from the lifting properties of the classes under the map of "forgetting" a point labeled by $m = 0$ and from the grading. Finally, one obtains the structure of a Frobenius manifold on $(\mathscr{H}^{(r)}, \eta)$ because $\Phi_0(\mathbf{x})$ is a polynomial. We refer the interested reader to [16] for details.

Remark 5.6 Strictly speaking, the state space of the KdV$_r$ theory should be $(\tilde{\mathscr{H}}^{(r)}, \tilde{\eta})$, where $\tilde{\mathscr{H}}^{(r)}$ is an r-dimensional vector space with basis $\{e_0, \ldots, e_{r-1}\}$, and with a metric in this basis given by $\tilde{\eta}_{m_1, m_2} := 1$ if $m_1 + m_2 \equiv (r-2) \mod r$ and 0 otherwise. However, there is an obvious orthogonal decomposition of vector spaces $\tilde{\mathscr{H}}^{(r)} = \mathscr{H}^{(r)} \oplus \mathscr{H}'$ where $\mathscr{H}'$ is the one dimensional vector space with basis $\{e_{r-1}\}$, and because $c^{1/r}$ vanishes on $\overline{\mathscr{M}}^{1/r, \mathbf{m}}_{g,n}$ whenever one of the m_i is $r-1$. Therefore, the decomposition $\tilde{\mathscr{H}}^{(r)} = \mathscr{H}^{(r)} \oplus \mathscr{H}'$ is a direct sum of CohFTs where $\mathscr{H}'$ is the trivial, one-dimensional CohFT. For this reason, we can and will restrict ourselves to the state space $(\mathscr{H}^{(r)}, \eta)$.

What is interesting about this approach is that it offers the possibility of generalization to higher genera. In [16], axioms were formulated for the virtual class $c^{1/r}$ in order to obtain a CohFT in all genera drawing upon an analogy with Gromov-Witten invariants. These axioms should be viewed as an analog of those of Behrend-Manin [2] for the virtual fundamental class in the case of the moduli space of stable maps.

Finally, it is also worth observing that the Frobenius structure here *cannot* arise as the quantum cohomology of a smooth, projective variety because it would correspond to fractional dimensional cohomology classes on the target variety.

6. The tensor product of Frobenius manifolds

The category of cohomological field theories has a natural tensor product [23] described as follows.

Definition 6.1 *Let $(\mathscr{H}', \eta', \Lambda')$ and $(\mathscr{H}'', \eta'', \Lambda'')$ be CohFTs. Their tensor product is $(\mathscr{H}' \otimes \mathscr{H}'', \eta' \otimes \eta'', \Lambda)$ where*

$$\Lambda_{g,n}(v'_1 \otimes v''_1, \ldots, v'_n \otimes v''_n) := \Lambda'_{g,n}(v'_1, \ldots v'_n) \cup \Lambda''_{g,n}(v''_1, \ldots, v''_n)$$

for all v' in $\mathscr{H}'$ and v'' in $\mathscr{H}''$.

This is nothing more than the fact that the diagonal map $\overline{\mathscr{M}}_{g,n} \to \overline{\mathscr{M}}_{g,n} \times \overline{\mathscr{M}}_{g,n}$ is a coproduct with respect to the composition maps of the modular operad $\{H_\bullet(\overline{\mathscr{M}}_{g,n})\}$. We will only discuss the situation where $g = 0$ and the potential functions are polynomial in the flat coordinates.

The tensor product operation, when written in terms of the underlying potential functions, is highly nontrivial [18, 19], even in genus zero. In the case of Gromov-Witten invariants, Behrend [1] proved that the CohFT arising from $\overline{\mathscr{M}}_{g,n}(V' \times V'')$ is the tensor product of that arising from $\overline{\mathscr{M}}_{g,n}(V')$ and $\overline{\mathscr{M}}_{g,n}(V'')$. When restricted to genus zero, one can view this result as a deformation of the Künneth theorem.

There exists a notion of the tensor product of (nonformal) Frobenius manifolds [18]. Because the potential functions of KdV_r are polynomial, their tensor product is also a (nonformal) Frobenius manifold. There is, however, an unanswered question. Is there a natural moduli space whose intersection numbers assemble into a generating function related to the tensor products of the Frobenius manifolds associated to KdV_{r^i} for $i = 1 \ldots k$? These moduli spaces would be analogs of $\overline{\mathscr{M}}_{0,n}(V_1 \times \cdots \times V_k)$ in the theory of Gromov-Witten invariants. The answer turns out to be *yes*, as explained in the following theorem. This theorem is an immediate consequence of Lemma 4.13.

Theorem 6.2 *Let $(\mathscr{H}^{(r^i)}, \eta^{(i)})$ denote the Frobenius manifold associated to the r^i-th Gelfand-Dickey hierarchy, KdV_{r^i}, for $r^i \geq 2$ and $i = 1 \ldots, k$. Let $(\mathscr{H}, \eta)$ denote the tensor product of these Frobenius manifolds. The tensor product Frobenius structure arises from $\Lambda := \{\Lambda_{0,n}\}$ where the forms $\Lambda_{0,n} \in H^\bullet(\overline{\mathscr{M}}_{g,n}) \otimes T^n \mathscr{H}^*$, are defined by*

$$\Lambda_{0,n}(e_{m_1^1} \otimes \ldots \otimes e_{m_1^k}, \ldots, e_{m_n^1} \otimes \ldots \otimes e_{m_n^k}) = (\prod_{i=1}^{k} r^i) p_* c^{1/\mathbf{r}},$$

and where $p : \overline{\mathscr{M}}_{g,n}^{1/\mathbf{r},\bar{\mathbf{m}}} \to \overline{\mathscr{M}}_{g,n}$, $\mathbf{r} := (r^1, \ldots, r^k)$, $\bar{\mathbf{m}} := (\mathbf{m}^1, \ldots, \mathbf{m}^k)$, $m_j^i = 0, \ldots, r^i - 2$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, n$.

One may regard the above result as a geometric realization (a kind of "A-model") for the tensor product of these Frobenius manifolds. Isomorphic Frobenius manifolds can be constructed from versal deformations of A_{r^i-1} singularities in [24, 25] (a Landau-Ginzburg model). The latter construction can be generalized to yield a Frobenius manifold associated to any singularity of type A-D-E, or even more generally, to Coxeter groups. It is known that as Frobenius manifolds, the tensor product of the Frobenius manifolds associated to A_2 (KdV_3) and A_3 (KdV_4) is isomorphic to the Frobenius manifold associated to E_6, while the tensor product of the Frobenius manifolds associated to A_2 (KdV_3) and A_4 (KdV_5) is isomorphic to the Frobenius manifold associated to E_8 [4, 6].

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On Fedosov's approach to deformation quantization with separation of variables

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Abstract. The description of all deformation quantizations with separation of variables on a Kähler manifold from [8] is used to identify the Fedosov star-product of Wick type constructed by M. Bordemann and S. Waldmann in [3]. This star-product is shown to be the one with separation of variables which corresponds to the trivial deformation of the Kähler form in the sense of [8]. To this end a formal Fock bundle on a Kähler manifold is introduced and an associative multiplication on its sections is defined.

Keywords: deformation quantization, Fock bundle, Kähler manifold

Mathematics Subject Classifications (2000): 53D55, 53B35, 55R65

Introduction

For a given vector space E we call formal vectors the elements of the space $E[\nu^{-1}, \nu]]$ of formal Laurent series in a formal parameter ν with a finite principle part and coefficients in E. Thus we consider the field of formal numbers $\mathbf{K} = \mathbf{C}[\nu^{-1}, \nu]]$, formal functions, forms, and differential operators.

Deformation quantization of a Poisson manifold $(M, \{\cdot,\cdot\})$ defined by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in [2], is a structure of associative algebra on the space of formal functions $\mathscr{F} = C^\infty(M)[\nu^{-1}, \nu]]$. The product $*$ in this algebra (called a star-product) is a $\mathbf{K}$-linear ν-adically continuous product given on functions $f, g \in C^\infty(M)$ by the formula

$$f * g = \sum_{r=0}^{\infty} \nu^r C_r(f,g), \tag{1}$$

where C_r are bidifferential operators, $C_0(f,g) = fg$ and $C_1(f,g) - C_1(g,f) = i\{f,g\}$. The constant 1 is assumed to be the unit in the algebra $(\mathscr{F}, *)$.

Two star-products $*_1$ and $*_2$ are called equivalent if there exists an isomorphism of algebras $B : (\mathscr{F}, *_1) \to (\mathscr{F}, *_2)$ given by a formal differential operator $B = Id + \nu B_1 + \nu^2 B_2 + \dots$.

The problem of existence and classification up to equivalence of star-products on Poisson manifolds was first solved for symplectic manifolds (the main refer-

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ences are [5,6,7,12,13]; for a historical account see [14]). In the general case it was solved by Kontsevich [10].

Let M be a Kähler manifold, endowed with a Kähler $(1,1)$-form ω_{-1} and the corresponding Poisson bracket. In [8], we gave a simple geometric description of all star-products on M which have the following property of separation of variables: In a local holomorphic chart the operators C_r from (1) act on the first argument by antiholomorphic derivatives, and on the second argument by holomorphic ones. We have shown that these star-products are naturally parameterized by geometric objects, the formal deformations of the Kähler form $(1/\nu)\omega_{-1}$.

The interest in deformation quantization with separation of variables is explained by the fact that the Wick star-product on $\mathbf{C}^n$ and the star-products obtained from Berezin's quantization on Kähler manifolds in [4,11,9] have the property of separation of variables.

In [3], Bordemann and Waldmann constructed a star-product with separation of variables on an arbitrary Kähler manifold (M, ω_{-1}), using the geometric approach developed by Fedosov in [6,7]. The goal of this letter is to identify the star-product obtained in [3], using the parameterization from [8]. We show that this star-product corresponds to the trivial deformation of the Kähler form $(1/\nu)\omega_{-1}$.

1. Deformation quantizations with separation of variables

For an open subset $U \subset M$, set $\mathscr{F}(U) = C^\infty(U)[\nu^{-1}, \nu]]$. Since the star-product (1) is given by formal bidifferential operators, it can be localized to any open subset $U \subset M$. We denote its restriction to $\mathscr{F}(U)$ also by $*$.

Denote by $\mathscr{L}^*(U)$ and $\mathscr{R}^*(U)$ the sets of all operators of left and right star-multiplication in the algebra $(\mathscr{F}(U), *)$, respectively. All these operators are formal differential ones. The subalgebras $\mathscr{L}^*(U)$ and $\mathscr{R}^*(U)$ of the algebra of formal differential operators on U are commutants of each other.

Now let (M, ω_{-1}) be a Kähler manifold with the Kähler $(1,1)$-form ω_{-1}. Consider a star-product $*$ on M with the following property of separation of variables. For an arbitrary local coordinate chart $U \subset M$ with holomorphic coordinates $\{z^k\}$ (and antiholomorphic coordinates $\{\bar{z}^l\}$) assume that the operators from $\mathscr{L}^*(U)$ contain only holomorphic derivatives and the operators from $\mathscr{R}^*(U)$ contain only antiholomorphic ones. This is equivalent to the fact that the operators from $\mathscr{L}^*(U)$ and $\mathscr{R}^*(U)$ commute with the pointwise multiplication operators by antiholomorphic and holomorphic functions on U, respectively. It means that, given a holomorphic function a and antiholomorphic function b on U, the pointwise multiplication operators by a and b belong to $\mathscr{L}^*(U)$ and $\mathscr{R}^*(U)$, respectively. Therefore $L_a^* = a$ and $R_b^* = b$, so that for $f \in \mathscr{F}(U)$, $a * f = af$ and $f * b = bf$ hold. This property was used for the definition of quantization with separation of variables in [8].

It was shown in [8] that star-products with separation of variables on (M, ω_{-1}) are in 1-1 correspondence with formal deformations of the Kähler form $(1/\nu)\omega_{-1}$, i.e., with the formal forms $\omega = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \cdots$ such that all ω_r, $r \geq 0$, are closed but not necessarily non degenerate $(1,1)$-forms on M.

Given an arbitrary formal deformation ω of the Kähler form $(1/\nu)\omega_{-1}$, one can recover the corresponding star-product with separation of variables as follows. On each contractible coordinate chart $(U, \{z^k\})$ on M choose a formal potential $\Phi = (1/\nu)\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \cdots$ of the form ω, so that $\omega = i\partial\bar{\partial}\Phi$. Then $L^*_{\partial\Phi/\partial z^k} = \partial\Phi/\partial z^k + \partial/\partial z^k$ and $R^*_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \partial/\partial\bar{z}^l$. Moreover, the set $\mathscr{L}^*(U)$ consists of all formal differential operators which commute with all $R^*_{\bar{z}^l} = \bar{z}^l$ and $R^*_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \partial/\partial\bar{z}^l$, and, respectively, $\mathscr{R}^*(U)$ is the commutant of the set of all operators $L^*_{z^k} = z^k$ and $L^*_{\partial\Phi/\partial z^k}$. This completely determines the star-product.

Remark In [3], star-products with separation of variables on Kähler manifolds are called star-products of Wick type, since the Wick star-product is the simplest one of this kind. However, one can consider star-products with separation of variables on an arbitrary symplectic manifold endowed with a pair of transversal Lagrangian polarizations (see [1]). In the Kähler case these are the holomorphic and antiholomorphic polarizations.

2. The formal Wick algebras bundle and the formal Fock bundle

Consider $\mathbf{C}^n$ with holomorphic coordinates $\{\zeta^k\}$ (and antiholomorphic coordinates $\{\bar{\zeta}^l\}$) endowed with a Hermitian (1,1)-form $ig_{kl}d\zeta^k \wedge d\bar{\zeta}^l$ (here g_{kl} are constants). Denote by $\circ$ the Wick star-product on $(\mathbf{C}^n, ig_{kl}d\zeta^k \wedge d\bar{\zeta}^l)$. This is the star-product with separation of variables, corresponding to the trivial deformation of the (1,1)-form $(1/\nu)ig_{kl}d\zeta^k \wedge d\bar{\zeta}^l$. The Wick star-product of functions $f, g \in C^\infty(\mathbf{C}^n)$ is given by the well-known explicit formula

$$f \circ g = \sum_{r=0}^{\infty} \frac{\nu^r}{r!} g^{l_1 k_1} \ldots g^{l_r k_r} \frac{\partial^r f}{\partial\bar{\zeta}^{l_1} \ldots \partial\bar{\zeta}^{l_r}} \frac{\partial^r g}{\partial\zeta^{k_1} \ldots \partial\zeta^{k_r}},$$

where (g^{lk}) is the inverse matrix of (g_{kl}). Here, as well as in the rest of the paper, we shall use Einstein's summation convention.

We now introduce the following gradings on the variables ν, ζ^k, and $\bar{\zeta}^l$: $\deg_\nu(\nu) = 1$, $\deg_\nu(\zeta) = \deg_\nu(\bar{\zeta}) = 0$; $\deg'_s(\zeta) = 1$, $\deg'_s(\nu) = \deg'_s(\bar{\zeta}) = 0$; $\deg''_s(\bar{\zeta}) = 1$, $\deg''_s(\nu) = \deg''_s(\zeta) = 0$; $\deg_s = \deg'_s + \deg''_s$; $\mathrm{Deg}' = \deg_\nu + \deg'_s$; $\mathrm{Deg}'' = \deg_\nu + \deg''_s$; $\mathrm{Deg} = \mathrm{Deg}' + \mathrm{Deg}'' = 2\deg_\nu + \deg_s$.

The Wick product $\circ$ is a graded product on polynomials in ν, ζ^k, and $\bar{\zeta}^l$ with respect to the gradings Deg', Deg'', and Deg. The total grading Deg is analogous to the one on the formal Weyl algebra used by Fedosov.

The "normal ordering" procedure establishes a 1-1 correspondence between the polynomials in $\mathbf{K}[\zeta^k, \bar{\zeta}^l]$ and holomorphic differential operators on $\mathbf{C}^n$ with coefficients in $\mathbf{K}[\zeta^k]$. Set $\hat{\zeta}^k = \zeta^k$ and $\hat{\bar{\zeta}}^l = \nu g^{lk}\partial/\partial\zeta^k$. The "normal ordering" associates to a polynomial $\phi(\zeta, \bar{\zeta}) = \phi_{\alpha,\beta}\zeta^\alpha\bar{\zeta}^\beta$ the operator $\hat{\phi} = \phi_{\alpha,\beta}\hat{\zeta}^\alpha\hat{\bar{\zeta}}^\beta$.

Here $\alpha = (k_1, \ldots, k_p)$ and $\beta = (l_1, \ldots, l_q)$ are multi-indices, $\zeta^\alpha = \zeta^{k_1} \cdots \zeta^{k_p}$, $\bar{\zeta}^\beta = \bar{\zeta}^{l_1} \cdots \bar{\zeta}^{l_q}$, $\hat{\zeta}^\alpha = \hat{\zeta}^{k_1} \cdots \hat{\zeta}^{k_p}$, $\hat{\bar{\zeta}}^\beta = \hat{\bar{\zeta}}^{l_1} \cdots \hat{\bar{\zeta}}^{l_q}$, and $\phi_{\alpha,\beta} \in \mathbf{K}$ is symmetric with respect to α and β separately. The polynomial ϕ is called the Wick symbol of the operator $\hat{\phi}$. The operator product transferred to Wick symbols provides the Wick product $\circ$.

The Wick product $\circ$ can be extended to the space W of formal series in ν^{-1}, ν, ζ^k, and $\bar{\zeta}^l$ with a finite principal part in ν,

$$w = \sum_{\substack{r \geq r_0 \\ p,q \geq 0}} \nu^r \sum_{\substack{\alpha,\beta \\ |\alpha|=p, |\beta|=q}} w_{r,\alpha,\beta} \zeta^\alpha \bar{\zeta}^\beta .$$

Here r_0 belongs to $\mathbf{Z}$, $\alpha = (k_1, \ldots, k_p)$ and $\beta = (l_1, \ldots, l_q)$ are multi-indices, $\zeta^\alpha = \zeta^{k_1} \cdots \zeta^{k_p}$, $\bar{\zeta}^\beta = \bar{\zeta}^{l_1} \cdots \bar{\zeta}^{l_q}$, and the terms of the series are ordered by increasing degrees $\mathrm{Deg} = p + q + 2r$. The obtained algebra $(W, \circ)$ is called a *formal Wick algebra.*

A *formal Fock space* V on $\mathbf{C}^n$ is the subspace of W of formal series in ν and ζ^k, i.e., of the formal series $v = \sum_{r \geq r_0, \alpha} \nu^r v_{r,\alpha} \zeta^\alpha$. Denote by $\bar{V}$ the subspace of W of formal series in ν and $\bar{\zeta}^l$.

Consider the following projection operators in W, $\Pi' w = w|_{\bar{\zeta}=0}$, $\Pi'' w = w|_{\zeta=0}$, and $\Pi w = w|_{\zeta=\bar{\zeta}=0}$, $w \in W$. Then $\Pi' W = V$, $\Pi'' W = \bar{V}$, and $\Pi W = \mathbf{K}$.

The kernels of the projections Π' and Π'' consist of the formal series $w \in W$ with all the terms containing at least one antiholomorphic variable $\bar{\zeta}^l$ or a holomorphic variable ζ^k, respectively. It is easy to check that $Ker\ \Pi'$ and $Ker\ \Pi''$ are respectively left- and right-ideals in the Wick algebra $(W, \circ)$. It follows, in particular, that $Ran\ \Pi' = V \cong W / Ker\ \Pi'$ is a left W-module. An element $w \in W$ acts on V by a formal holomorphic differential operator T_w on $\mathbf{C}^n$ given by the formula $T_w v = \Pi'(w \circ v)$, $v \in V$. One can show that if $w \in \mathbf{K}[\zeta, \bar{\zeta}]$ then $T_w = \hat{w}$, i.e., T_w is the differential operator with the Wick symbol w. We shall say for general $w \in W$ that w is the Wick symbol of T_w and denote $T_w = \hat{w}$. It is easy to check that the mapping $W \ni w \mapsto \hat{w}$ is an injective homomorphism of the algebra $(W, \circ)$ to the algebra of formal differential operators on $\mathbf{C}^n$.

Lemma 1 *For $w \in W$, $\Pi' w = 0$ iff the operator $\hat{w}$ annihilates the subspace of formal constants $\mathbf{K} \subset V$, and $\Pi'' w = 0$ iff Ran $\hat{w} \subset$ Ker Π.*

The proof of the lemma follows from elementary properties of Wick symbols.

Given a Kähler manifold (M, ω_{-1}) of complex dimension $dim_{\mathbf{C}} M = n$, consider the unions of the formal Wick algebras and of the formal Fock spaces associated to each tangent space to M. Thus we obtain the bundles of formal Wick algebras $\mathbf{W}$ and of formal Fock spaces $\mathbf{V}$ on M. For an open subset $U \subset M$, denote by $\mathscr{W}(U)$ and $\mathscr{V}(U)$ the spaces of local sections of $\mathbf{W}$ and $\mathbf{V}$ on U, respectively. Set $\mathscr{W} = \mathscr{W}(M)$ and $\mathscr{V} = \mathscr{V}(M)$.

On a coordinate chart $(U, \{z^k\})$ on M, we introduce the following gradings on the 1-forms dz^k and $d\bar{z}^l$: $\deg'_a(dz) = \deg''_a(d\bar{z}) = 1$, $\deg'_a(d\bar{z}) = \deg''_a(dz) = 0$;

$\deg_a = \deg' a + \deg''_a$. Denote by $\Lambda = \oplus_r \Lambda^r$, the $\deg_a$-graded algebra of differential forms on M.

There exist natural inclusions of the spaces $\mathscr{F} \otimes \Lambda \subset \mathscr{V} \otimes \Lambda \subset \mathscr{W} \otimes \Lambda$ of the (formal) scalar, **V**- and **W**-valued differential forms on M respectively (the tensor product is taken over $C^\infty(M)$, $\otimes = \otimes_{C^\infty(M)}$).

The fiberwise Wick product and the action of W on V in the first factor of the tensor product together with the wedge-product of differential forms in the second factor define the structures of $\deg_a$-graded algebra on $\mathscr{W} \otimes \Lambda$ and of its $\deg_a$-graded module on $\mathscr{V} \otimes \Lambda$. The product in $\mathscr{W} \otimes \Lambda$ will also be denoted by $\circ$. The projections Π, Π', and Π'' define fiberwise projections in $\mathscr{W} \otimes \Lambda$ denoted by the same symbols. The action of an element $w \in \mathscr{W} \otimes \Lambda$ on the space $\mathscr{V} \otimes \Lambda$ is given by the operator $\hat{w}$ defined, as above, by the expression $\hat{w}v = \Pi'(w \circ v)$, where $v \in \mathscr{V} \otimes \Lambda$. We have $\Pi'(\mathscr{W} \otimes \Lambda) = \mathscr{V} \otimes \Lambda$ and $\Pi(\mathscr{W} \otimes \Lambda) = \mathscr{F} \otimes \Lambda$.

In the sequel we shall always denote by ζ^k and $\bar{\zeta}^l$ the fiber coordinates on the tangent bundle TM in the frame $\{\partial/\partial z^k, \partial/\partial \bar{z}^l\}$ on a coordinate chart $(U, \{z^k\})$ on M.

Notice that for a local section $w(z,\bar{z}) = \sum_{r \geq r_0, \alpha, \beta} \nu^r w_{r,\alpha,\beta}(z,\bar{z}) \zeta^\alpha \bar{\zeta}^\beta \in \mathscr{W}(U)$ the coefficients $w_{r,\alpha,\beta}(z,\bar{z})$ are covariant tensor fields on M, symmetric with respect to α and β separately.

3. Fedosov star-product of Wick type

We recall the construction by Bordemann and Waldmann of the Fedosov star-product of Wick type on a Kähler manifold (M, ω_{-1}) from [3]. (We use, however, different conventions and notations.)

Let ∇ denote the standard Kähler connection on M. It can be naturally extended to tensors, and thus to the bundles **W** and **V**. For technical reasons it will be convenient to denote its extension to **W** also by ∇, and its extension to **V** by $\hat{\nabla}$.

Express the Kähler form ω_{-1} on M and the Kähler connection ∇ on $\mathscr{W} \otimes \Lambda$ in local coordinates $\{z^k, \bar{z}^l, \zeta^k, \bar{\zeta}^l\}$ by:

$$\omega_{-1} = i g_{kl} dz^k \wedge d\bar{z}^l, \quad \nabla = d - \Gamma^s_{ki} \zeta^i (\partial/\partial \zeta^s) dz^k - \Gamma^t_{lj} \bar{\zeta}^j (\partial/\partial \bar{\zeta}^t) d\bar{z}^l,$$

where $\Gamma^s_{ki} = g^{ls} \partial g_{kl} / \partial z^i$ and $\Gamma^t_{lj} = g^{kt} \partial g_{kl} / \partial \bar{z}^j$ are the Christoffel symbols and (g^{lk}) is the inverse matrix of(g_{kl}). Then $\hat{\nabla} = d - \Gamma^s_{ki} \zeta^i (\partial/\partial \zeta^s) dz^k$.

Introduce an element $R \in \mathscr{W} \otimes \Lambda^2$ such that it is given in local coordinates $\{z^k, \bar{z}^l, \zeta^k, \bar{\zeta}^l\}$ by the formula $R = (-g^{ts} \partial g_{kt} \wedge \bar{\partial} g_{sl} + \partial \bar{\partial} g_{kl}) \zeta^k \bar{\zeta}^l$.

The curvature of the connection ∇ on the bundle **W** was calculated in [3]: $\nabla^2 = (1/\nu) ad_{Wick}(R)$. A straightforward calculation leads to the following result.

Lemma 2 *The curvature of the connection* $\hat{\nabla}$ *on the bundle* **V** *is expressed via* R *as follows,* $\hat{\nabla}^2 = (1/\nu)\hat{R}$.

Introduce Fedosov's operators δ and δ^{-1} on $\mathscr{W} \otimes \Lambda$. In local coordinates $\delta = (\partial/\partial \zeta^k) dz^k + (\partial/\partial \bar{\zeta}^l) d\bar{z}^l$ and the operator δ^{-1} is defined as follows. For an element $a \in \mathscr{W} \otimes \Lambda^q$ such that $\deg_s = p$ set $\delta^{-1}a = 0$ if $p+q=0$ and $\delta^{-1}a = (p+q)^{-1}(\zeta^k i(\partial/\partial z^k) + \bar{\zeta}^l i(\partial/\partial \bar{z}^l))a$ if $p+q > 0$.

Then $\delta = (1/\nu)ad_{Wick}(\vartheta)$, where $\vartheta = g_{kl}\bar{\zeta}^l dz^k - g_{kl}\zeta^k d\bar{z}^l$ (see [3]).

It was shown in [3] that there exists a unique element $r \in \mathscr{W} \otimes \Lambda^1$ which satisfies the equations $\delta^{-1}\ r = 0$ and $\delta r = R + \nabla r + (1/\nu) r \circ r$, and contains only non-negative powers of ν.

In [3], a flat Fedosov's connection D on $\mathbf{W}$ is defined as follows, $D = -\delta + \nabla + (1/\nu)ad_{Wick}(r)$. It is a $\deg_a$-graded derivation of the algebra $(\mathscr{W} \otimes \Lambda, \circ)$. Therefore $\mathscr{W}_D = Ker\ D \cap \mathscr{W}$ is closed under Wick multiplication.

It was proved in [3] that the mapping $\Pi : \mathscr{W}_D \to \mathscr{F}$ is, in fact, a bijection. By transferring the product from the Fedosov algebra $(\mathscr{W}_D, \circ)$ to $\mathscr{F}$ via this bijection, one obtains a star-product $*$ on (M, ω_{-1}). Moreover, it was proved in [3] that $*$ is a star-product with separation of variables. The proof was based on the following important statement (Lemma 4.5 in [3]): $r \in Ker\ \Pi' \cap Ker\ \Pi''$, i.e., in any local expression of r each term contains the variables ζ^k and $\bar{\zeta}^l$ for some indices k, l. We reformulate this statement using Lemma 1.

Lemma 3 *The operator $\hat{r}$ in $\mathscr{V}$ annihilates the subspace $\mathscr{F} \otimes \Lambda \subset \mathscr{V} \otimes \Lambda$. In particular, $\hat{r}1 = 0$. Moreover, Ran $\hat{r} \subset Ker\ \Pi$.*

We are going to show that the star-product with separation of variables $*$ constructed in [3] corresponds to the trivial deformation $\omega = (1/\nu)\omega_{-1}$ of the Kähler form $(1/\nu)\omega_{-1}$.

4. The Fock algebra

Using the fact that $\delta = (1/\nu)ad_{Wick}(\vartheta)$, one can express D as follows, $D = \nabla + (1/\nu)ad_{Wick}(\gamma)$, where $\gamma = -\vartheta + r$.

Introduce a connection $\hat{D}$ on $\mathbf{V}$ by the formula $\hat{D} = \hat{\nabla} + (1/\nu)\hat{\gamma}$.

One can split the connections $\nabla, D, \hat{D}$, the operator δ, and the element r into the sums of their (1,0)- and (0,1)-components: $\nabla = \nabla' + \nabla''$, $D = D' + D''$, $\hat{D} = \hat{D}' + \hat{D}''$, $\delta = \delta' + \delta''$, $r = r' + r''$. In local coordinates denote $\nabla_k = \nabla_{\partial/\partial z^k}$, $\nabla_l = \nabla_{\partial/\partial \bar{z}^l}$, so that $\nabla' = \nabla_k dz^k$, $\nabla'' = \nabla_l d\bar{z}^l$. In a similar fashion, we introduce D_k, D_l, $\hat{D}_k$, and $\hat{D}_l$. Let $r = r_k dz^k + r_l d\bar{z}^l$ be a local expression of the element r. Then $r' = r_k dz^k$ and $r'' = r_l d\bar{z}^l$.

A simple calculation shows that $(1/\nu)\hat{\vartheta} = \partial/\partial\zeta^k dz^k - \eta_l d\bar{z}^l$, where $\eta_l = (1/\nu)g_{kl}\zeta^k$. Therefore,

$$\hat{D}_k = \partial/\partial z^k - \partial/\partial\zeta^k + (1/\nu)\hat{r}_k, \quad \hat{D}_l = \partial/\partial \bar{z}^l + \eta_l + (1/\nu)\hat{r}_l. \tag{2}$$

Lemma 4 *Let $f \in \mathscr{F}(U)$, where $(U, \{z^k\})$ is a coordinate chart on M. Then $\hat{D}_k f = \partial f/\partial z^k$. In particular, $\hat{D}_k 1 = 0$.*

The lemma trivially follows from Lemma 3 and Formula (2).

Lemma 5 *For* $w \in \mathscr{W} \otimes \Lambda$ *one has* $[\hat{\nabla}, \hat{w}] = \widehat{\nabla w}$.

Here, as well as below, the commutator is the $\deg_a$-graded commutator in the graded algebra of endomorphisms of $\mathscr{V} \otimes \Lambda$.

The lemma is an easy consequence of the fact that ∇ is a $\deg_a$-graded derivation of the algebra $(\mathscr{W} \otimes \Lambda, \circ)$. It implies the following proposition.

Proposition 1 *For* $w \in \mathscr{W} \otimes \Lambda$ *the formula* $[\hat{D}, \hat{w}] = \widehat{Dw}$ *holds.*

Let us write $\omega = (1/\nu)\omega_{-1}$.

Lemma 6

(i) $[\hat{\nabla}, \hat{\vartheta}] = 0$;

(ii) $(1/\nu)[\hat{\vartheta}, \hat{r}] = \widehat{\delta r}$;

(iii) $\hat{\vartheta}^2 = i\nu^2\omega$.

The Lemma is proved by straightforward calculations. It also implies the following proposition.

Proposition 2 *The connection* $\hat{D}$ *on* $\mathbf{V}$ *has a scalar curvature,* $\hat{D}^2 = i\omega$.

The subspace $\mathscr{W}_{D''} = \mathit{Ker}\ D'' \cap \mathscr{W}$ of the algebra $(\mathscr{W}, \circ)$ is closed under the Wick product. We shall use the algebra $(\mathscr{W}_{D''}, \circ)$ to define a product on the space $\mathscr{V}$.

Introduce Fedosov's operator δ''^{-1} on $\mathscr{W} \otimes \Lambda$ defining it in the local coordinates on a chart $(U, \{z^k\})$ as follows. Let $w \in (\mathscr{W} \otimes \Lambda)(U)$ be such that $\deg''_s(w) = p$, $\deg''_a(w) = q$. Set $\delta''^{-1} a = 0$ if $p + q = 0$ and $\delta''^{-1} a = (p+q)^{-1} \bar{\zeta}^l i(\partial/\partial \bar{z}^l) a$ if $p + q > 0$. Then for $w \in \mathscr{W} \otimes \Lambda$ one has $(\delta'' \delta''^{-1} + \delta''^{-1} \delta'') w = w - w_0$, where w_0 is the $(\deg''_s + \deg''_a)$-homogeneous component of w of the degree 0. For an element $w \in \mathscr{W} \otimes \Lambda$ denote by $w^{(q)}$ its Deg''-homogeneous component of the degree q.

The following proposition can be proved by Fedosov's technique developed in [6].

Proposition 3 *The mapping* $\Pi' : \mathscr{W}_{D''} \to \mathscr{V}$ *is a bijection. For an element* $v \in \mathscr{V}$ *such that* $\deg_\nu(v) = 0$ *(i.e., which does not depend on the formal parameter* ν*) the unique element* $w \in \mathscr{W}_{D''}$ *such that* $v = \Pi' w$ *can be calculated recursively with respect to the degree* Deg'' *by*

$$w^{(0)} = v;$$

$$w^{(q+1)} = \delta''^{-1}\left(\nabla'' w^{(q)} + (1/\nu) \sum_{p=0}^{q} ad_{Wick}(r''^{(p+1)}) w^{(q-p)}\right).$$

Denote by $\bullet$ the product in $\mathscr{V}$ obtained by pushing forward the product in the algebra $(\mathscr{W}_{D''}, \circ)$ by the mapping Π'. Thus we obtain a *Fock algebra* $(\mathscr{V}, \bullet)$. For $v \in \mathscr{V}$ denote by $L^\bullet_v$ and $R^\bullet_v$ the operators of left and right multiplication by v in the algebra $(\mathscr{V}, \bullet)$, respectively. Set $\mathscr{L}^\bullet = \{L^\bullet_v | v \in \mathscr{V}\}$ and $\mathscr{R}^\bullet = \{R^\bullet_v | v \in \mathscr{V}\}$.

Lemma 7 *For $w \in \mathscr{W}_{D''}$, the operator $\hat{w}$ coincides with the left multiplication operator by the element $v = \Pi' w$ in the Fock algebra $(\mathscr{V}, \bullet)$, $\hat{w} = L^{\bullet}_{v}$.*

Proof. For $w_1, w_2 \in \mathscr{W}_{D''}$, set $v_1 = \Pi' w_1$ and $v_2 = \Pi' w_2$. Then, by definition, $v_1 \bullet v_2 = \Pi'(w_1 \circ w_2)$. Since Π' is a projection, $w_2 - v_2 \in Ker\, \Pi'$. Taking into account that $Ker\, \Pi'$ is a left-ideal in the algebra $(\mathscr{W}, \circ)$, we get $w_1 \circ (w_2 - v_2) \in Ker\, \Pi'$. Therefore $\Pi'(w_1 \circ w_2) = \Pi'(w_1 \circ v_2) = \hat{w}_1 v_2$, whence the Lemma follows. ∎

Since the action of the operators $\hat{w}$, $w \in \mathscr{W}$, on $\mathscr{V}$ is fiberwise, it follows from Lemma 7 that the operator of pointwise multiplication by $f \in \mathscr{F}$ (also denoted by f) commutes with all operators from $\mathscr{L}^{\bullet}$. Therefore, $f \in \mathscr{R}^{\bullet}$, namely, $R^{\bullet}_{f} = f$.

Fix a coordinate chart $(U, \{z^k\})$ on M.

Lemma 8 $R^{\bullet}_{\eta_l} = \hat{D}_l$.

Proof. Let $w \in \mathscr{W}_{D''}(U)$, $v = \Pi' w \in \mathscr{V}(U)$. It follows from Lemma 7 and Proposition 1 that $[\hat{D}_l, L^{\bullet}_{v}] = [\hat{D}_l, \hat{w}] = \widehat{D_l w} = 0$, therefore $\hat{D}_l \in \mathscr{R}^{\bullet}$. Using Formula (2) and Lemma 3, we get $\hat{D}_l 1 = \eta_l$, whence $\hat{D}_l = R^{\bullet}_{\eta_l}$. ∎

Denote $\mathscr{U} = \Pi'(\mathscr{W}_D) \subset \mathscr{V}$. Since $\mathscr{W}_D \subset \mathscr{W}_{D''}$, and the projection Π' establishes an isomorphism between the algebras $(\mathscr{W}_{D''}, \circ)$ and $(\mathscr{V}, \bullet)$, the subspace $\mathscr{U} \subset \mathscr{V}$ is closed under multiplication $\bullet$ and the projection Π' maps the Fedosov algebra $(\mathscr{W}_D, \circ)$ isomorphically onto the subalgebra $(\mathscr{U}, \bullet)$ of the Fock algebra $(\mathscr{V}, \bullet)$.

Lemma 9 *For $w \in \mathscr{W}_{D''}(U)$ and $v = \Pi' w \in \mathscr{V}(U)$, one has $D_k w \in \mathscr{W}_{D''}(U)$ and $[\hat{D}_k, L^{\bullet}_{v}] = \widehat{D_k w} = L^{\bullet}_{\hat{D}_k v}$.*

Proof. Using Lemma 7 and Proposition 1, we obtain that $[\hat{D}_k, L^{\bullet}_{v}] = [\hat{D}_k, \hat{w}] = \widehat{D_k w}$. Since Fedosov's connection D is flat, $D^2 = 0$, we have that $[D_k, D_l] = 0$, whence $D_l D_k w = D_k D_l w = 0$, i.e., $D_k w \in \mathscr{W}_{D''}(U)$ and therefore $\widehat{D_k w} \in \mathscr{L}^{\bullet}(U)$. Using Lemma 4 we get $[\hat{D}_k, L^{\bullet}_{v}]1 = \hat{D}_k v - L^{\bullet}_{v} \hat{D}_k 1 = \hat{D}_k v$ and thus $\widehat{D_k w} = L^{\bullet}_{\hat{D}_k v}$, which concludes the proof. ∎

Denote by $\mathscr{V}_{\hat{D}'}(U) = Ker\, \hat{D}' \cap \mathscr{V}(U)$ the space of local sections of the Fock bundle $\mathbf{V}$ on an open subset $U \subset M$, annihilated by $\hat{D}'$. Set $\mathscr{V}_{\hat{D}'} = \mathscr{V}_{\hat{D}'}(M)$.

Proposition 4 $\mathscr{U} = \mathscr{V}_{\hat{D}'}$.

Proof. We have to show that on any coordinate chart $(U, \{z^k\})$ on M, $w \in \mathscr{W}_{D''}(U)$, and $v = \Pi' w \in \mathscr{V}(U)$ the condition $D_k w = 0$ holds iff $\hat{D}_k v = 0$. The assertion follows immediately from the equality $L^{\bullet}_{\hat{D}_k v} = \widehat{D_k w}$ proved in Lemma 9 and the fact that the mapping $\mathscr{W} \ni w \mapsto \hat{w}$ is injective. ∎

We can obtain the star-product $*$ on M from the algebra $(\mathcal{V}_{\hat{D}'}, \bullet) = (\mathcal{U}, \bullet)$. Let $v_1, v_2 \in \mathcal{V}_{\hat{D}'}$, $f_1 = \Pi v_1$, $f_2 = \Pi v_2 \in \mathcal{F}$. Then $f_1 * f_2 = \Pi(v_1 \bullet v_2)$.

Let Φ_{-1} be a local potential of the form $\omega_{-1} = i g_{kl} dz^k \wedge d\bar{z}^l$ on a coordinate chart $(U, \{z^k\})$ on M, so that $\partial^2 \Phi_{-1}/\partial z^k \partial \bar{z}^l = g_{kl}$. Then $\Phi = (1/\nu)\Phi_{-1}$ is a local potential of the form $\omega = (1/\nu)\omega_{-1}$. Set $Q_l = \partial\Phi/\partial\bar{z}^l + \eta_l$.

Proposition 5 $Q_l \in \mathcal{V}_{\hat{D}'}(U)$.

Proof. Using Lemma 4, we get that $\hat{D}_k \partial\Phi/\partial\bar{z}^l = \partial^2\Phi/\partial z^k \partial\bar{z}^l = (1/\nu) g_{kl}$. It follows from Proposition 2 that $[\hat{D}_l, \hat{D}_k] = (1/\nu) g_{kl}$. Now, $\hat{D}_k \eta_l = \hat{D}_k \hat{D}_l 1 = \hat{D}_l \hat{D}_k 1 - (1/\nu) g_{kl} = -(1/\nu) g_{kl}$ and therefore $\hat{D}' Q_l = 0$. ■

Since $*$ is known to be a star-product with separation of variables, then $R^*_{\bar{z}^l} = \bar{z}^l$ holds. This can be checked also directly. It follows from Lemma 4 that $\hat{D}_k \bar{z}^l = 0$, i.e., $\bar{z}^l \in \mathcal{V}_{\hat{D}'}(U)$. Let $v \in \mathcal{V}_{\hat{D}'}(U)$ and $f = \Pi v \in \mathcal{F}(U)$. Now $f * \bar{z}^l = \Pi(v \bullet \bar{z}^l) = \Pi(v\bar{z}^l) = f\bar{z}^l$, which proves the assertion.

In order to identify the star-product with separation of variables $*$ it remains to calculate $R^*_{\partial\Phi/\partial\bar{z}^l}$. Let $v \in \mathcal{V}_{\hat{D}'}(U)$ and $f = \Pi v$ as above. Let us first calculate $\Pi\hat{D}_l v$. Using Formula (2) we get $\Pi\hat{D}_l v = \Pi(\partial v/\partial\bar{z}^l + \eta_l v + (1/\nu)\hat{r}_l v)$. Since $\Pi\eta_l = 0$, we have $\Pi(\eta_l v) = 0$. Lemma 3 implies that $\Pi(\hat{r}_l v) = 0$. Finally we obtain that $\Pi\hat{D}_l v = \partial f/\partial\bar{z}^l$.

Since $\Pi Q_l = \partial\Phi/\partial\bar{z}^l$, we have that $f * \partial\Phi/\partial\bar{z}^l = \Pi(v \bullet Q_l) = \Pi(R^\bullet_{Q_l} v) = \Pi((\partial\Phi/\partial\bar{z}^l + \hat{D}_l)v) = (\partial\Phi/\partial\bar{z}^l + \partial/\partial\bar{z}^l) f$. Therefore $R^*_{\partial\Phi/\partial\bar{z}^l} = \partial\Phi/\partial\bar{z}^l + \partial/\partial\bar{z}^l$. Thus we have proved the desired result:

Theorem *The Fedosov star-product of Wick type $*$ on a Kähler manifold (M, ω_{-1}) is the star-product with separation of variables corresponding to the trivial deformation of the form $(1/\nu)\omega_{-1}$.*

One might also try to generalize the construction by Bordemann and Waldmann to obtain arbitrary deformation quantizations with separation of variables.

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The Master thesis of Moshé Flato

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Abstract. The genesis and impact of the M.Sc. thesis by Moshé Flato is analyzed. In this connection, the fruitful passage of Moshé in Lyon, *capitale mondiale de la gastronomie*, is evoked. Finally, some basic elements for a model in crystal-field theory are given as an important step on the way opened by Moshé.

Keywords: finite subgroups of $SU(2)$, Wigner-Racah algebra for finite and compact groups, crystal-field theory, molecular and condensed matter spectroscopy

Mathematics Subject Classifications (2000): 20C35, 81R05, 82D25, 81V80

To my teacher and friend Moshé Flato

Geneses

Moshé Flato prepared his M.Sc. thesis under the guidance of Giulio Racah at the Physics Department of the Hebrew University in Jerusalem at the end of the fifties [1]. The final examination took place in 1960. The genesis of the thesis can be described as follows.

From 1942 to 1949, Racah published a series of four major papers on new algebraic and group theoretical methods for the analysis of complex spectra in atomic and nuclear spectroscopy [2]. In particular, he developed the concepts of *irreducible tensor operator*, *coefficient of fractional parentage* (a concept which goes back to Goudsmit and Bacher) and *seniority number*. Furthermore, he showed how the use of a chain of continuous groups ending with $SO(3) \supset SO(2)$ makes it possible to calculate the coefficients of fractional parentage and to classify the state vectors originating from an ℓ^N configuration in spherical symmetry. In 1956, he became very interested in crystal-field theory (a theory for partly-filled shell ions embedded in crystals, solutions or complexes) on the occasion of discussions and seminars with Wily Low from the Microwave Division of the Department of Physics of the Hebrew University [3]. Crystal-field theory, also referred to as ligand-field theory, may be considered as a branch of quasi-atomic physics in the sense that this theory is concerned with the electronic properties of a central ion (with a configuration of type ℓ^N where $\ell = d$ or f) in interaction with its immediate surrounding. Racah realized that the simultaneous consideration of finite groups (for describing the ion in its environment) and continuous groups (for

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labeling the Russell-Saunders terms $[S,L]$ or multiplets $[J]$ of the ion) can be very fruitful in crystal-field theory. W. Low and G. Racah suggested to two students, Miss G. Schoenfeld and Mr. G. Rosengarten, to calculate the Coulomb, spin-orbit, and crystal-field matrices for d^N ions in cubical symmetries. The cubical symmetry is often an idealized (or first-order) symmetry so that G. Racah asked Mr. M. Flato to reconsider the problem of d^N ions in finite symmetry and to treat the case of trigonal and tetragonal symmetries which are of interest in absorption spectroscopy and in electron spin resonance spectroscopy.

Schoenfeld calculated the matrices for the d^2 and d^3 configurations in cubical symmetry (symmetry group $G = O$) and Rosengarten obtained the matrices for the d^4 and d^5 configurations in cubical symmetry too. Flato dealt with the case where distortions (i.e., *deformations*) of trigonal symmetry (symmetry group $G = D_3$) or tetragonal symmetry (symmetry group $G = D_4$) are introduced in order to describe a greater number of physical situations. Along these lines, he calculated the crystal-field matrices for the d^2 and d^3 configurations in tetragonal and trigonal symmetries.

From the group-theoretical point of view, the problem addressed by Schoenfeld and Rosengarten, and in a more complete way by Flato, concerns the labeling of the state vectors arising from the d^N configuration via the use of a chain of groups starting from the classification group $U(5)$ and ending with the point symmetry group G or its spinor group G^* with $G \sim G^*/Z_2$. The head group $U(5)$ corresponds to the fact that the $2d+1 = 5$ orbitals associated to the d-shell (for which $\ell = 2$) may undergo a unitary transformation without changing the spectrum of the relevant Hamiltonian. The end group is G (the geometrical symmetry group for the ion in its environment) or G^* (the spectral group that labels the spin, called the double group of G in the terminology of H. A. Bethe) depending on the parity, even or odd, of the number of electrons N. Therefore, Flato used the chains $U(5) \supset SO(3) \supset O \supset D_3$ or D_4, and $U(5) \supset SU(2) \supset O^* \supset D_3^*$ or D_4^*, for the configurations d^2 and d^3, respectively.

One of the advantages in using such chains of groups is that the Coulomb and spin-orbit matrices already calculated for the free ion can be easily transferred to the case of the ion in its surrounding. Indeed, the chain $U(5) \supset SO(3)$ or $U(5) \supset SU(2)$ is useful for the construction of the Coulomb and spin-orbit matrices. From a qualitative point of view, the chains $SO(3) \supset O \supset D_3$ or D_4, and $SU(2) \supset O^* \supset D_3^*$ or D_4^*, serve to describe the (external) *symmetry breaking* (arising from an inhomogeneous Stark effect) when going from the free ion to the ion in its environment. From a quantitative point of view, the calculation of the crystal-field matrix can be done via the Wigner-Eckart theorem for the group $SU(2)$ in a nonstandard basis, namely a basis adapted to the chain $SO(3) \supset O \supset D_3$ or D_4 for the d^2 configuration and the chain $SU(2) \supset O^* \supset D_3^*$ or D_4^* for the d^3 configuration.

To fully understand the originality of the works by Flato, one has to realize that at this time most of the works on crystal- and ligand-field theories for d^N ions in finite symmetries (involving both an idealized or high symmetry and a distortion or low symmetry) were conducted according to the approach by Tanabe, Sugano, and Kamimura (from the Japanese school of M. Kotani) [4, 5] and the one by Griffith [6, 7]. In these approaches, the end group, i.e., the point symmetry group G or its spinor group G^*, is considered as an isolated group so that the subduction (or descent in symmetry) $SO(3) \supset H \supset G$ or $SU(2) \supset H^* \supset G^*$ is not fully taken into account (H stands for a high symmetry group, like O, and G for a low symmetry group, like D_3): The information about the descent in symmetry $H \supset G$ or $H^* \supset G^*$ is kept at the minimal level and the link with $SO(3)$ or $SU(2)$ is not completely exploited. Therefore, in the Griffith [6, 7] and Tanabe, Sugano, and Kamimura [4, 5] approaches, all the necessary matrix elements must be calculated through the use of the Wigner-Racah algebra of G or G^*. As a consequence, the calculation of the crystal-field energy matrix is very easy but the calculation of the Coulomb and spin-orbit energy matrices is very complicated. As a matter of fact, the invariance of the Coulomb and spin-orbit interactions under $SO(3)$, when translated in terms of $H \supset G$, leads to sums of invariant irreducible tensor operators which are difficult to handle via the Wigner-Eckart theorem for the group G or G^*. This difficulty does not appear in the approach followed by Schoenfeld, Rosengarten, and Flato.

The M.Sc. work by Flato [1] constituted the first systematic attempt to combine the simplifications afforded by the various chains of groups of the type $U(5) \supset SO(3) \supset H \supset G$ and $U(5) \supset SU(2) \supset H^* \supset G^*$ for dealing with partly-filled shell ions in discrete symmetry. The works by Schoenfeld, Rosengarten, and Flato were used by Low and several of his collaborators for understanding optical spectra and electron paramagnetic resonance spectra of transition-metal ions in crystals (for example, see [3,8–10]). Although Moshé presented his M.Sc. thesis dissertation in 1960, several years later the subject was still up to date since he published a part of his thesis in 1965 [11].

The 1965 Flato paper and some impulse given by Moshé opened an avenue of new investigations in: (i) the Wigner-Racah algebra of the group $SU(2)$ in bases adapted to chains of the type $SU(2) \supset G^*$ where G^* is a group or a chain of groups of interest in molecular physics and condensed matter physics and (ii) the development of models in optical spectroscopy (mainly luminescence spectroscopy and photoemission spectroscopy) of d^N and f^N ions in crystalline fields (see [12–20] and references therein). Along this vein, in 1966 Moshé Flato suggested to a student, Maurice Kibler, from the 'Faculté des Sciences de l'Université de Lyon', to extend his M.Sc. thesis work to the situation where a magnetic interaction (arising from the Zeeman effect) is added to the Coulomb + spin-orbit + crystal-field Hamiltonian. This suggestion led the present author to develop the Wigner-Racah algebra of an arbitrary finite group embedded in $SU(2)$ and to construct a phenomenological model, where symmetry considerations play a central role, for ions in a crystal subjected to an external magnetic field [12, 15]. The research in that direction continued during several years with some students and collaborators of Kibler, especially in:

- the study of specific chains of groups,

- the application of symmetry adaptation to electron paramagnetic resonance,
- the unification of crystal-field and ligand-field theories,
- the development of models for photo-electron spectroscopy,
- the derivation of a phenomenological approach to multi-photon electronic spectroscopy.

In 1995, with the advent of strong technological progresses in two-photon absorption spectroscopy of rare earth ions doped materials, part of the above mentioned works were revived in view of the elaboration of a model for analyzing intensities of two-photon transitions between states of opposite parities for an ℓ^N ion in finite symmetry [20].

The rest of this paper is devoted to some personal recollections concerning the passage of Moshé in Lyon and to a brief description (in the appendix) of the basic ingredients for symmetry adaptation techniques in molecular and condensed matter spectroscopy developed by the present author more than thirty years ago under the direction of Moshé.

Moshé in Lyon

The arrival of Moshé in Lyon in 1964 has been a great event for many people. Although Moshé did not yet have a 'Doctorat d'État' (he submitted his 'thèse de doctorat ès sciences physiques' at 'la Sorbonne' in Paris on June 15, 1965), he joined the university staff of the 'Université de Lyon' as an associated professor in Physics under the recommendation of Jean Braconnier, 'doyen de la Faculté des Sciences de Lyon'. (Jean Braconnier had been a member of the Bourbaki group.) He kept his position of associated professor for three years, sharing his life between Lyon and Paris. From 1964 to 1967, it was a real pleasure to enjoy the presence in Lyon of Moshé and Daniel (Daniel Sternheimer), almost every Monday and Tuesday to the best of my recollection. During the academic year 1964–65, Moshé gave master lectures on group theory both for finite groups and continuous groups. For most people in the audience the way to treat the subject was quite new (discussion at two levels: mathematical exposure and illustration from many branches of physics) and the lectures were very successful. The next two academic years, Moshé undertook to give a complete panorama of modern physics starting from classical mechanics up to quantum field theory (with intermediate steps in relativistic mechanics, quantum mechanics and statistical mechanics). The corresponding lectures were given by Moshé with an extraordinary economy principle in the presentation, a high level content and a constant preoccupation of the understanding by the audience.

During this short period (1964–67), several students (D. Allouche, S. Ekong, and J. Sternheimer) achieved their 'thèse de doctorat de spécialité (troisième cycle)' and two other students (J. Gréa and M. Kibler) started to work on their 'thèse d'État' under the supervision of Moshé.

Many of us in Lyon (students, collaborators, and friends) have greatly benefited from Moshé not only for a transmission of a high level knowledge but also for having made possible to share with him a part of his dynamism, his enthusiasm, his love of physics (following Christian Frønsdal, one can add he was a physicist and a mathematician, teaching mathematics and having a passion for physics) and his respect of others. We also learned from Moshé how to be demanding for ourselves. His generosity (in all aspects of life) and his so particular way to consider each of us as a unique component of the world are qualities which should inspire us all.

Appendix: Finite symmetry adaptation

Let G^* be a (finite) subgroup of $SU(2)$. We use Γ $(= \Gamma_0, \Gamma_1, \Gamma_2, \ldots)$ to denote an irreducible representation class (IRC) of G^*, Γ_0 being the identity IRC of G^*. We write $[\Gamma]$ for the dimension of a matrix representation associated to Γ. Such a matrix representation can be spanned by vectors of type

$$|ja\Gamma\gamma) := \sum_{m=-j}^{j} |jm)\,(jm|ja\Gamma\gamma) \qquad (1)$$

which are linear combinations of vectors $|jm)$ adapted to the chain $SU(2) \supset U(1)$. In Equation (1), the branching label a has to be used when the IRC Γ of G^* occurs several times in the IRC (j) of $SU(2)$; this external multiplicity label may be described, at least partially, by IRC's of subgroups of $SU(2)$ that contain in turn the group G^*. Furthermore, the label γ is necessary when $[\Gamma] > 1$; it may be described sometimes by IRC's of subgroups of G^*. Equation (1) is of a group theoretical nature: The reduction coefficients $(jm|ja\Gamma\gamma)$ depend on the chain $SU(2) \supset G^*$ but not on Physics. They may be chosen, thanks to Schur's lemma, in such a way that the matrix representation associated to Γ is the same for all j's and a's. Equation (1) defines the $\{ja\Gamma\gamma\}$ scheme for the chain $SU(2) \supset G^*$ that is more appropriate in molecular or condensed matter spectroscopy than the $\{jm\}$ scheme for the chain $SU(2) \supset U(1)$. For j, a, and Γ fixed, the set $\{|ja\Gamma\gamma) \,:\, \gamma = 1, \ldots, [\Gamma]\}$ is a G^*-irreducible tensorial set of vectors associated to Γ. Similarly, from the $SU(2) \supset U(1)$ spherical tensor operators $U^{(k)}_q$, we define the operators

$$U^{(k)}_{a\Gamma\gamma} := \sum_{q=-k}^{k} U^{(k)}_q\,(kq|ka\Gamma\gamma) \qquad (2)$$

so that, for k, a and Γ fixed, the set $\left\{U^{(k)}_{a\Gamma\gamma} | \gamma = 1, \ldots, [\Gamma]\right\}$ is a G^*-irreducible tensorial set of operators associated to Γ. The latter G^*-irreducible tensorial sets are also labeled by IRC's of $SU(2)$ and, therefore, we can easily generate, by direct sum, nonstandard $SU(2)$-irreducible tensorial sets. Thus, we may apply the Wigner-Eckart theorem for the group $SU(2)$ in a nonstandard basis adapted to its subgroup

G^*. As a result, we have

$$(\tau_1 j_1 a_1 \Gamma_1 \gamma_1 | U^{(k)}_{a\Gamma\gamma} | \tau_2 j_2 a_2 \Gamma_2 \gamma_2) =$$

$$(\tau_1 j_1 \| U^{(k)} \| \tau_2 j_2) \sum_{a_3\Gamma_3\gamma_3} \begin{pmatrix} & j_1 & \\ a_3\Gamma_3\gamma_3 & & a_1\Gamma_1\gamma_1 \end{pmatrix} \overline{f} \begin{pmatrix} j_1 & k & j_2 \\ a_3\Gamma_3\gamma_3 & a\Gamma\gamma & a_2\Gamma_2\gamma_2 \end{pmatrix}^* \tag{3}$$

with

$$\overline{f} \begin{pmatrix} j_1 & j_2 & j_3 \\ a_1\Gamma_1\gamma_1 & a_2\Gamma_2\gamma_2 & a_3\Gamma_3\gamma_3 \end{pmatrix} := \tag{4}$$

$$\sum_{m_1 m_2 m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} (j_1 m_1 | j_1 a_1 \Gamma_1 \gamma_1)^* (j_2 m_2 | j_2 a_2 \Gamma_2 \gamma_2)^* (j_3 m_3 | j_3 a_3 \Gamma_3 \gamma_3)^*,$$

and

$$\begin{pmatrix} & j & \\ a_1\Gamma_1\gamma_1 & & a_2\Gamma_2\gamma_2 \end{pmatrix} := \sqrt{2j+1}\,\overline{f} \begin{pmatrix} j & 0 & j \\ a_1\Gamma_1\gamma_1 & a_0\Gamma_0\gamma_0 & a_2\Gamma_2\gamma_2 \end{pmatrix}. \tag{5}$$

In the right-hand side of (3), the quantum numbers τ_1 and τ_2, external to the chain $SU(2) \supset G^*$ (they depend on the physics of the considered problem), appear only in the reduced matrix element $(\tau_1 j_1 \| U^{(k)} \| \tau_2 j_2)$.

The $\overline{f}$ or 3-$ja\Gamma\gamma$ symbol in (4) is an $SU(2) \supset G^*$ symmetry adapted form of the usual 3-jm Wigner symbol. It constitutes a symmetrized form of the coefficient [12]

$$f \begin{pmatrix} j_1 & j_2 & k \\ a_1\Gamma_1\gamma_1 & a_2\Gamma_2\gamma_2 & a\Gamma\gamma \end{pmatrix} := \tag{6}$$

$$\sum_{m_1 q m_2} (-1)^{j_1 - m_1} \begin{pmatrix} j_1 & k & j_2 \\ -m_1 & q & m_2 \end{pmatrix} (j_1 m_1 | j_1 a_1 \Gamma_1 \gamma_1)^* (kq | ka\Gamma\gamma) (j_2 m_2 | j_2 a_2 \Gamma_2 \gamma_2)$$

that generalizes the f coefficient defined via

$$(J\Gamma\gamma | U^{(k)}_{\Gamma_0} | J'\Gamma'\gamma') := \delta(\Gamma', \Gamma)\, \delta(\gamma', \gamma)\, f \begin{pmatrix} J & J' & k \\ \Gamma & \Gamma & \Gamma_0 \end{pmatrix} \tag{7}$$

in Moshé's M.Sc. thesis [1].

The 2-$ja\Gamma\gamma$ symbol defined by (5) turns out to be an $SU(2) \supset G^*$ symmetry adapted form of the Herring-Wigner metric tensor whose spherical components may be taken as

$$\begin{pmatrix} & j & \\ m_1 & & m_2 \end{pmatrix} := (-1)^{j+m_1}\,\delta(m_2,-m_1). \tag{8}$$

From equations (5) and (8) we have

$$\begin{pmatrix} & j & \\ a_1\Gamma_1\gamma_1 & & a_2\Gamma_2\gamma_2 \end{pmatrix} = \sum_m (-1)^{j+m}\,(jm|ja_1\Gamma_1\gamma_1)^*\,(j,-m|ja_2\Gamma_2\gamma_2)^*. \tag{9}$$

The metric tensor given by (9) allows us to handle all the phases occurring in the $\{ja\Gamma\gamma\}$ scheme.

By combining (4), (6), and (9), we can rewrite (3) in the simple form

$$(\tau_1 j_1 a_1\Gamma_1\gamma_1|U^{(k)}_{a\Gamma\gamma}|\tau_2 j_2 a_2\Gamma_2\gamma_2) = \\ (\tau_1 j_1\|U^{(k)}\|\tau_2 j_2)\, f\begin{pmatrix} j_1 & j_2 & k \\ a_1\Gamma_1\gamma_1 & a_2\Gamma_2\gamma_2 & a\Gamma\gamma \end{pmatrix}. \tag{10}$$

The interest of (3) and (10) for electronic spectroscopy of ions in crystalline fields has been discussed at length in [12, Part II]. From a mathematical viewpoint, it is to be observed that the factorization in (10) into the product of a G^*-dependent factor (the f coefficient) by a G^*-independent factor (the reduced matrix element) is valid whether the group G^* is multiplicity free or not. The internal multiplicity problem arising when G^* is not multiplicity free is thus easily solved by making use of (10).

An important property of the f coefficient is given by the following factorization formula which arises as a consequence of a lemma by Racah [2, Part IV]. Indeed, we have

$$f\begin{pmatrix} j_1 & j_2 & k \\ a_1\Gamma_1\gamma_1 & a_2\Gamma_2\gamma_2 & a\Gamma\gamma \end{pmatrix} = \\ (-1)^{2k}\,\frac{1}{\sqrt{2j_1+1}}\sum_\beta (j_2a_2\Gamma_2+ka\Gamma|j_1a_1\beta\Gamma_1)^*(\Gamma_2\Gamma\gamma_2\gamma|\Gamma_2\Gamma\beta\Gamma_1\gamma_1)^*, \tag{11}$$

where $(j_2a_2\Gamma_2+ka\Gamma|j_1a_1\beta\Gamma_1)$ is an isoscalar factor or reduced f symbol (independent of the labels γ_1, γ_2, and γ) and $(\Gamma_2\Gamma\gamma_2\gamma|\Gamma_2\Gamma\beta\Gamma_1\gamma_1)$ a Clebsch-Gordan coefficient for the group G^* considered as an isolated entity. The label β is an internal multiplicity label to be used when the Kronecker product $\Gamma_2 \otimes \Gamma$ is not multiplicity free.

Equations (10) and (11) are of central importance in the study of magnetic and electronic properties of an ℓ^N ion in G (with $G \sim G^*/Z_2$) symmetry as well as in rotational spectroscopy of molecules invariant under G. They proved to be useful in the development of phenomenological models for dealing with both energy levels and intensities of transitions. The reader may consult [12–14] for an application to crystal-field theory.

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Anticipative Chen-Souriau cohomology and Hochschild cohomology

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Abstract. We define random forms over the based loop space of a manifold respectively to the set of semi-martingales and we show that the stochastic cohomology is equal to the Hochschild cohomology.

Keywords: loop space, diffeology, semi-martingales, Hochschild cohomology

Mathematics Subject Classifications (2000): 55N20, 58A10, 58G32

1. Introduction

Let M be a compact manifold. Let $L_x(M)$ be the based loop space of M. There are two stochastic cohomology theories of $L_x(M)$.

1) In the first one, we suppose that the manifold is Riemannian. We endow the loop space $L_x(M)$ with the Brownian bridge measure. We consider on a stochastic loop (which is only continuous), a Hilbert tangent space, which was introduced in a preliminary form by Bismut [5] and under a final form by Jones and Léandre [18]. This allows us to define Sobolev spaces of functionals [20, 21] and Sobolev spaces of forms on the Brownian bridge. The stochastic exterior derivative acts continuously on the intersection of all Sobolev spaces of forms. This allows to define some Sobolev stochastic cohomology groups of the based loop space $L_x(M)$ [22, 23, 24]. Since there is no Sobolev imbedding theorem in infinite dimension, the space of smooth forms (those which belong to all of the Sobolev spaces) is constituted of forms which are only almost surely defined. It is the purpose of the Malliavin calculus to do analysis over objects which are almost surely defined [34]. If we consider smooth loops, the Hochschild complex constitutes an algebraic model of the loop space: If the manifold is simply connected, the de Rham cohomology groups of $L_x(M)$ are equal to the Hochschild cohomology groups [1, 11, 16]. It is the purpose of the works of Léandre [23, 24] to show that this classical result of algebraic topology remains true for the Sobolev cohomology of the loop space. References [23] and [24] furnish a stochastic generalization of the classical result of Adams [1] and Chen [11], for the Sobolev cohomology of the loop space. Therefore, the Sobolev cohomology is equal to the deterministic de Rham cohomology of the smooth loop space.

2) The second stochastic cohomology theory deals with objects which are only almost surely defined too, but without to give a (Sobolev) number to a random

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form. The main idea goes back to the considerations of Chen [11], Souriau [36], and Iglesias [17] in the deterministic case. These authors consider a family of plots valued in the loop space (or another complicated space), which satisfy a system of axioms: This family constitutes a diffeology. A form over the loop space is seen through a plot: The problem to define an exterior derivative becomes trivial, because the exterior derivative is seen by using this system of plots. A stochastic generalization was performed in [25, 26, 27, 28]. There are two different stochastic diffeologies of the loop space: The first one uses the general theory of semi-martingales on a manifold which belong to all of the H^p [14]. The second one uses stochastic integrals with respect to a flat Brownian motion. The main interest of the second one is that stochastic integrals are in fact Hölderian (and not only continuous), and that there are partitions of unity over the Hölderian loop space [2, 3]. We work for the second stochastic diffeology over the Hölderian loop space, and we can apply the traditional tools of de Rham cohomology theory, which are strongly based upon the existence of partitions of unity. This allows to show that the stochastic cohomology of Chen-Souriau with respect of the diffeology involved with stochastic integrals is equal to the deterministic cohomology of the Hölderian loop space [26, 28]. We give in Section 2 an example of a form which is a form with respect of the diffeology involved with stochastic integrals and not with the general diffeology of semi-martingales which are in all of the H^p. (See [14].)

The purpose of this work is to compute the stochastic cohomology, when we consider a set of general semi-martingales as diffeology: We work over the continuous loop space, where there is no partition of unity in general. The interest to consider more general processes is the following: In algebraic topology, the natural object is not a manifold but a $C-W$ complex. People in algebraic topology consider the loop space of a $C-W$ complex which is a notion which is more general than the loop space of a manifold. A beginning of theory of random processes over an Euclidean complex was performed by Brin and Kifer in [8]. In these random processes the Walsh martingale plays an important role, which is not a stochastic integral. So the eventual generalization of stochastic homology of a $C-W$ complex needs a more general class of random processes than stochastic integrals. With respect to this larger class of diffeology, in [27] it was already shown that the stochastic cohomology groups are invariant under differentiable homotopies. The stochastic cohomology in this sense gives an invariant under differentiable homotopies, although we are not able to compute explicitely these cohomology groups; namely, the traditional tools for computing the de Rham cohomology are inefficients.

But, if the manifold is simply connected, we know that the cohomology of the based smooth loop space is equal to the Hochschild cohomology. If we can show that the stochastic cohomology with respect to the diffeology involved with general semi-martingales is equal to the Hochschild cohomology, we will then be able to show as a consequence that the stochastic cohomology is equal to the cohomology of the smooth loop space.

The fact that the Hochschild cohomology for a simply connected manifold is equal to the cohomology of the smooth based loop space uses a spectral sequence because the based path space is a fibration over the manifold with fiber the loop space. For the Sobolev cohomology, this last property remains true by using Driver's flow [15, 13, 24]. The work in [23] uses the quasi-invariance formulas of Albeverio-Hoegh-Krohn in order to show this property for the loop space of a homogeneous manifold.

In order to get this fibration property for this stochastic diffeology, we have to impose that the semi-martingales depend, in some sense, on the future, which give a little bit richer diffeology than the diffeology of [25].

This leads us to give a diffeological calculus with anticipating conditions. With respect to this system of anticipative plots, we get a stochastic cohomology theory which is equal to the Hochschild cohomology, therefore to the cohomology of the smooth based loop space. For this we suppose that the manifold *M* is simply connected.

Let us motivate the study of stochastic cohomology: Equivariant cohomology of the loop space is related to the finite-dimensional index theory [6, 7, 16]. There are series of forms over the loop space, for instance the Bismut-Chern character, and it is interesting to know if they converge in some convenient functional spaces. Moreover, the infinite-dimensional index theory is related to a theory of bundles over the loop space [19, 33, 12, 10, 37]. The simplest case of such a bundle is constituted by line bundles over the loop space [9], which is given by its curvature, if the loop space is simply connected. It becomes therefore interesting in order to speak of Hilbert spaces of sections of bundles to consider stochastic cohomology groups [29, 30, 26, 28]. In a particular case, we can consider a Kac-Moody group: Over the underlying loop group we can construct the Brownian bridge measure or another measure. For instance, we can perform a change of time over the loop group such that the typical loops are no longer Hölderian. We can define the Hilbert space of functionals over the Kac-Moody group, but in the curvature of the circle bundle of the Kac-Moody group, some stochastic integrals appear. This shows us that in the transition functional some stochastic integrals appear, where we have to deform the typical loop in the group into a constant loop. By this deformation, we get laws which are not quasi-invariant with respect of the original mesure for the non-Hölderian loops in the group. If we make a change of time over the loop group, the theory of measures over the Kac-Moody group implies an understanding of stochastic forms (which are almost surely defined) over the set of semi-martingales (see [31]).

2. Statement of the conjecture

Let M be a compact Riemannian manifold of dimension d. We embed it into the Euclidean space $R^{d'}$.

Let (Ω, F_t, P) be a filtered probability space with a filtration F_t satisfying the usual conditions. A semi-martingale on M can be seen as a semi-martingale on $R^{d'}$ constrained to be on M.

On semi-martingales from $[0,1]$ on $R^{d'}$, starting from a given deterministic point on $R^{d'}$ and which are in all of the H^p (see [14]), we construct a convenient topology: If $X_t = M_t + V_t$ is a semi-martingale where M_t is a local martingale and V_t an adapted continuous finite variational process, we put

$$\|X_\bullet\|_p^p = E[< M, M >_1^{p/2}] + E[(\int_0^1 |dV_s|)^p]. \tag{1}$$

We consider over the set of semi-martingales the Fréchet topology endowed with the family of norms $\|\cdot\|_p$. We call $S_{\infty-}$ the Fréchet space which is obtained from this procedure.

Definition 1 *A stochastic plot $\psi = (U, \phi_i, Z_i, \Omega_i)$ of dimension n from an open subset U of R^n is given by the following data:*

i) A countable partition of Ω in measurable sets Ω_i.

ii) A smooth map $\phi_i : U \times R^m \to S_{\infty-}$ for any m. This implies that $\phi_i(u,v)_\bullet$ has almost surely a smooth version in (u,v) defined for the uniform topology over the set of paths in $R^{d'}$.

iii) A family of random variables $Z_i(u)$ from Ω into R^m which depend smoothly on u for $u \in U$ in all of the L^p. We choose a smooth version of this family by the Kolmogorov lemma [35].

iv) $\phi_i(u, Z_i(\omega)(u))(\omega)_\bullet$ belongs almost surely to $L_x(M)$ if $\omega \in \Omega_i$ for the considered smooth version in (u,v) of $\phi_i(u,v)(\omega)_\bullet$.

The system of stochastic plots ψ constitutes a diffeology of $L_x(M)$. We identify two stochastic plots $(U, \phi_i^1, Z_i^1, \Omega_i^1)$ and $(U, \phi_i^2, Z_i^2, \Omega_i^2)$ if $\phi_i^1(\cdot, Z_i^1(\cdot))_\bullet = \phi_j^2(\cdot, Z_j^2(\cdot))_\bullet$ almost surely in $\Omega_i^1 \cap \Omega_j^2$ as smooth maps from U into the based loop space endowed with the uniform topology.

Definition 2 *A stochastic form σ_{st} over $L_x(M)$ is given by the following data:*

i) To each stochastic plot $\psi = (U, \phi_i, Z_i, \Omega_i)$, we associate almost surely a random smooth form over U $\psi^*\sigma_{st}$, for almost all ω belonging to each Ω_i.

Moreover, the system of random finite-dimensional forms $\psi^*\sigma_{st}$ check the following consistency properties:

ii) Let $j : U_1 \to U_2$ be a smooth deterministic map. Let $\psi^2 = (U_2, \phi_i^2, Z_i^2, \Omega_i)$ be a stochastic plot. Let $\psi^1 = (U_1, \phi_i^1, Z_i^1, \Omega_i)$ be the composite plot:

$$\phi_i^1(u_1, v) = \phi_i^2(j \circ u_1, v), \tag{2}$$

and $Z_i^1(u_1) = Z_i^2(j \circ u_1)$ almost surely as random smooth maps. Then almost surely, $\psi^{1*}\sigma_{st} = j^*\psi^{2*}\sigma_{st}$.

iii) Let $\psi^1 = (U, \phi_i^1, Z_i^1, \Omega_i^1)$ and $\psi^2 = (U, \phi_j^2, Z_j^2, \Omega_j^2)$ be two stochastic plots. Suppose that there exist i, j, and a measurable transformation $F : \Omega_i^1 \to \Omega_j^2$ defined over a set of measure strictly larger than 0 such that almost surely:

$$\phi_i^1(\cdot, Z_i^1(\omega)(\cdot))(\omega). = \phi_j^2(\cdot, Z_j^2(F\omega)(\cdot))(F\omega). \tag{3}$$

Then almost surely

$$\psi^{1*}\sigma_{st}(\omega) = \psi^{2*}\sigma_{st}(F(\omega)). \tag{4}$$

Remark We will not make precise the technicalities which arise in condition iii), because we take smooth versions of $\phi_1^1(u, Z_i^1(\omega)(u))(\omega)$, $\phi_j^2(\cdot, Z_j^2(\omega)(u))(\omega)$, $\psi^{1*}\sigma_{st}(\omega)$, and $\psi^{2*}\sigma_{st}(\omega)$.

Let us give an example of stochastic forms over $L_x(M)$ with respect to this stochastic diffeology. Let us introduce

$$\tilde{\omega} = \omega_1 \otimes \cdots \otimes \omega_n \tag{5}$$

an element of $\Omega_{\bullet}(M)^{\otimes n}$, where $\Omega_{\bullet}(M)$ is the set of smooth forms of degree larger than 1 over the manifold M. Let us consider the Chen form

$$\sigma_{st}(\tilde{\omega}) = \int_{0<s_1<\cdots<s_n<1} \omega_1(d\gamma_{s_1}, \cdot) \wedge \cdots \wedge \omega_n(d\gamma_{s_n}, \cdot). \tag{6}$$

We can extend the forms ω_i into the forms $\omega_{i,e}$ on the Euclidean space where the manifold is imbedded. We can consider the stochastic form on $U \times R^m$ defined for a plot ψ by

$$\sigma_{U\times R^m}(v)(\tilde{\omega}) = \int_{0<s_1<\cdots<s_n<1} \omega_{1,e}(d\phi_i(u,v)_{s_1}, \partial\phi_i(u,v)_{s_1}) \wedge \cdots \wedge \omega_{n,e}(d\phi_i(u,v)_{s_n}, \partial\phi_i(u,v)_{s_n}). \tag{7}$$

We take the derivative of $\phi_i(u,v)_s$ in the direction of $u \in U$ and $v \in R^m$. We get by this procedure a random form over $U \times R^m$. Let us explain a little more this formula. Suppose that $\omega_{i,e}$ is of degree r_i, such that $\sigma_{U\times R^m}(v)(\tilde{\omega})$ is of degree $N = \sum(r_i - 1)$. We consider vector fields over $U \times R^m$, $X_1, \ldots, X_N$, we apply $\omega_{i,e}(d\phi_i(u,v)_{s_i}, \cdot)$ to the $(r_i - 1)$-uple $\partial_{X_j}\phi_i(u,v)_{s_i}$ where j runs from $r_1 - 1 + r_2 - 1 + \cdots r_{i-1}$ to $r_1 - 1 + r_2 - 1 + \cdots r_i - 1$, and we perform the stochastic integral over the simplex. This gives by the Kolmogorov lemma [35] a random smooth multilinear expression over $U \times R^m$, which is partially antisymmetric. It remains to antisymmetrize the expression because we have wedge products in $\omega_{i,e}(d\phi_i(u,v)_{s_i}, \cdot)$. We have the random map from U into $U \times R^m$ denoted by $\tilde{Z}_i$: $u \to (u, Z_i(u))$. We get a random smooth form over $U \times R^m$. On Ω_i, we put

$$\psi^* \sigma_{st} = \tilde{Z}_i^* \sigma_{U \times R^m}(\tilde{\omega}). \tag{8}$$

$\psi^* \sigma_{st}$ checks clearly the hypothesis of compatibility of Definition 2 ii) because the Stratonovitch integrals are purely local. We denote by $\Sigma(\tilde{\omega})$ the stochastic Chen iterated integral σ_{st}, and the map stochastic Chen iterated integral is denote by Σ.

Let us define the Hochschild boundary:

$$\begin{aligned} b(\tilde{\omega}) = & -\sum(-1)^{\varepsilon i-1} \omega_1 \otimes \cdots \otimes d\omega_i \otimes \cdots \otimes \omega_n \\ & -\sum(-1)^{\varepsilon_i} \omega_1 \otimes \cdots \otimes \omega_i \wedge \omega_{i+1} \otimes \cdots \otimes \omega_n, \end{aligned} \tag{9}$$

where $\varepsilon_i = \sum_{j<i} deg \omega_j - 1$.

We can define the exterior derivative of a stochastic form σ_{st} by putting for a stochastic plot ψ:

$$\psi^* d\sigma_{st} = d\psi^* \sigma_{st}. \tag{10}$$

By the same proof of [25, p. 22], we get:

$$d\sigma_{st}(\tilde{\omega}) = \sigma_{st}(b\tilde{\omega}). \tag{11}$$

The map Σ: $\tilde{\omega} \to \sigma_{st}(\tilde{\omega})$ has a kernel [16]. It is defined as follows. Let us consider a smooth function $f : M \to R$. Let $S_i(f)$ be the following operation: Between ω_i and ω_{i+1}, we intercalate f in the tensor product which constitutes $\tilde{\omega}$. We get the expression $\omega_1 \otimes \omega_2 \otimes \cdots \otimes \omega_i \otimes f \otimes \omega_{i+1} \otimes \cdots \otimes \omega_n$. The kernel of the map stochastic Chen iterated integrals Σ is constituted of finite linear combinations $[b, S_i(f)]\tilde{\omega}$, b being the Hochschild boundary.

$H_{\bullet}H_{\bullet}(R, \Omega_{\bullet}(M), R)$ is the cohomology group for the Hochschild boundary of the Hochschild complex, when we consider a **finite** sum of elementary tensor products $\tilde{\omega}$, with different length of tensor products, modulo the kernel of the map stochastic Chen iterated integrals.

$H^*_{st}(L_x(M))$ is the cohomology group for the stochastic exterior derivative defined on stochastic differential forms by (10).

The main result of this work is the following:

Conjecture*Let us suppose M simply connected. The map stochastic Chen iterated integrals induces an isomorphism between* $H_{\bullet}H_{\bullet}(R, \Omega_{\bullet}(M), R)$ *and* $H^*_{st}(L_x(M))$.

As we shall see, the proof of this conjecture presents a difficulty. In the next part, we will prove a statement which is very close to this conjecture.

Remark We give an example in a simple case of a 0-form (that is a functional) for the diffeology associated to stochastic integral which has no chance to be extended to a form over the set of semi-martingale which belong to all of the H^p. We work over R and we consider as diffeology the set of $\phi(u)_t = \int_0^t A_s(u)\delta B_s$ where B_s is a flat one-dimensional Brownian motion and $A_s(u)$ depends smoothly in all of the L^p from u belonging to the parameter space U which is an open subset of R^n. σ_{st} is defined as follows: $\phi^*\sigma_{st} = \int_0^1 A_s(u)ds$ and constitutes an example of a functional for our diffeology which seems difficult to extend to more general semi-martingales in R.

3. Proof of the approximated theorem of the conjecture

We recall that the based path space $P_x(M)$ is the set of continuous maps $\gamma_\bullet$ from $[0,1]$ into M such that $\gamma_0 = x$. We define a stochastic plot of the path space.

Definition 3 *A stochastic plot $\psi = (U, \phi_i, Z_i(u), \Omega_i)$ for an open subset U of R^n of the based path space is given by the following data:*

i) A countable partition of Ω in measurable sets Ω_i.

ii) A smooth map $\phi_i : U \times R^m \to S_{\infty-}$. This implies that $\phi_i(u,v)_\bullet$ has almost surely a smooth version in (u,v) defined for the uniform topology over the set of paths in $R^{d'}$.

iii) A family of random variables $Z_i(u)$ from Ω into R^m depending smoothly on $u \in U$ in L^p. The Kolmogorov lemma implies that there exists a smooth version of $Z_i(u)$ [35].

iv) $\phi_i(u, Z_i(\omega)(u))(\omega)_\bullet$ belongs almost surely to $P_x(M)$ if $\omega \in \Omega_i$ for the considered smooth version in (u,v) of $\phi_i(u,v)(\omega)$.

We consider the same identification of two stochastic plots with values in $P_x(M)$ as the equivalence relation stated after Definition 1. We have an analogous definition of a stochastic form over $P_x(M)$ along the lines of Definition 2.

We remark that a stochastic plot over $L_x(M)$ defines a stochastic plot over $P_x(M)$. Therefore, by restriction, a stochastic form over $P_x(M)$ defines a stochastic form over $L_x(M)$.

The exterior derivative of a stochastic form over $P_x(M)$ is defined as in (10).

One of the main ingredient of the proof of the main theorem is the following, which reflects that the based path space is contractible:

Theorem 1 $H^*_{st}(P_x(M)) = R$.

Proof. Let us extend the Levi-Cività connection over M, which we suppose endowed with a Riemannian metric, to a connection over $R^{d'}$. We get the notion of parallel transport over a stochastic plot $\phi_i(u,v)_\bullet$: We note it $\tau_i(u,v)_\bullet$. The main property of $\tau_i(u,v)_\bullet$ is the following: If $\omega \in \Omega_i$, $\tau_i(u, Z_i(\omega)(u))(\omega)_\bullet$ defines a map from $T_x(M)$ onto $T_{\phi_i(u,Z_i(\omega)(u))(\omega)_\bullet}(M)$, which is a rotation, because we have extended the Levi-Cevità connection from M to $R^{d'}$ in a metric connection, and because on Ω_i, $\phi_i(u,Z_i(\omega)(u))(\omega)_\cdot$ is an element of $P_x(M)$. We note by $\xi_i(u,v)_\cdot$ the semi-martingale in $R^{d'}$ defined in the sense of Stratonovitch by:

$$d\xi_i(u,v)(\omega)_s = \tau_i^{-1}(u,v)_s d\phi_i(u,v)_s \tag{12}$$

and which starts at 0. If $\omega \in \Omega_i$, $\xi_i(u,Z_i(\omega))(\omega)_\cdot$ is a path starting at 0 in $T_x(M)$. We consider $\varepsilon \in [0,1]$ and the path $\varepsilon\xi_i(u,v)(\omega)_s = \xi_i(\varepsilon,u,v)(\omega)_s$. We introduce the Stratonovitch differential equation:

$$d\phi_i(\varepsilon,u,v)(\omega)_s = \tau_i(\varepsilon,u,v)_s d\xi_i(\varepsilon,u,v)(\omega)_s \tag{13}$$

starting at x. $\tau_i(\varepsilon,u,v)_s$ is the parallel transport over the random path which is realized by the semi-martingale $\phi_i(\varepsilon,u,v)(\omega)_\cdot$. Moreover, we can suppose that the Christoffel symbols of the extended connection on $R^{d'}$ are bounded, so that $\phi_i(\varepsilon,u,v)$ depends smoothly on (ε,u,v) in $S_{\infty-}$. Moreover, if $\omega \in \Omega_i$, $\phi_i(\varepsilon,u,Z_i(\omega)(u))(\omega)_\cdot$ is almost surely in $P_x(M)$. On the other hand, $\phi_i(0,u,Z_i(\omega)(u))(\omega)_s = x$ and $\phi_i(1,u,Z_i(\omega)(u))(\omega)_\cdot = \phi_i(u,Z_i(\omega)(u))(\omega)_\cdot$. In others words, we have deformed the plot ψ into the constant plot by remaining in the considered set of plots. We get a plot ψ^{ext} given by $(U \times [0,1], \phi_i(\varepsilon,u,v), Z_i, \Omega_i)$.

Let us consider a closed stochastic form σ_{st} of degree strictly larger than 0. $\psi^{ext,*}\sigma_{st}$ gives a random closed form on $U \times [0,1]$. We consider the retraction operation from $U \times [0,1]$ to U given by $g_\lambda(u,\varepsilon) = (u, f_\lambda(\varepsilon))$, $\lambda \in [0,1]$, for a convenient smooth function f_λ from $[0,1]$ into 0 which is equal to 0 at $\lambda = 0$. Since $\psi^{ext,*}\sigma_{st}$ is closed and since a form over a point vanishes if its degree is strictly larger than 0, we get by the Cartan formula:

$$\psi^{ext,*}\sigma_{st} = d\int_0^1 g_\lambda^* i_{\frac{\partial g_\lambda}{\partial \lambda}} \psi^{ext,*}\sigma_{st} d\lambda. \tag{14}$$

This implies by condition ii) of Definition 2 that

$$\psi^*\sigma_{st} = d_u \int_0^1 g_\lambda^* i_{\frac{\partial g_\lambda}{\partial \lambda}} \psi^{ext,*}\sigma_{st} d\lambda. \tag{15}$$

Therefore $\sigma_{st} = d\tilde{\sigma}_{st}$ if the degree of σ_{st} is strictly larger than 0. If this degree vanishes, we have by the same type of argument

$$\psi^*\sigma_{st} = C, \tag{16}$$

where C is almost surely constant by condition iii) of Definition 2, and corresponds to the value of σ_{st} associated to the constant stochastic plot. ■

Let us recall what is the Hochschild complex associated to the based path space. Let

$$\tilde{\omega} = \omega_1 \otimes \cdots \otimes \omega_n \tag{17}$$

be an element of the algebraic tensor product $\Omega_{\bullet}(M)^{\otimes n-1}\otimes\Omega(M)$, where $\Omega(M)$ denotes the set of smooth forms over M without any restriction upon the degree. Let us consider the Chen form:

$$\sigma_{st}(\tilde{\omega}) = \int_{0<s_1<\cdots<s_{n-1}<1} \omega_1(d\gamma_{s_1},\cdot)\wedge\cdots\wedge\omega_{n-1}(d\gamma_{s_{n-1}},\cdot)\wedge\omega_n. \tag{18}$$

We can extend the forms ω_i to forms $\omega_{i,e}$ on the Euclidean space where the manifold is imbedded. Let ψ be a stochastic plot over $P_x(M)$. Instead of (7), we consider the iterated integral between 1 and $n-1$ and we consider the wedge product by $\omega_{n,e}(\partial\phi_i(u,v)_1)$. $\sigma_{st}(\tilde{\omega})$ is a Chen form over the based path space of degree $\sum_0^{n-1}(\deg(\omega_i)-1)+\deg(\omega_n)$. Moreover, $d\sigma_{st}(\tilde{\omega}) = \sigma_{st}(b\tilde{\omega})$ where b is given by the same formula as in (9).

The kernel of the operation stochastic Chen iterated integral is given just after Equation (11). We get a complex, called the algebraic Hochschild complex, which operates over **finite sums**, and we denote its cohomology groups by $H_{\bullet}H_{\bullet}(R,\Omega_{\bullet}(M),\Omega(M))$.

Theorem 2 $H_{\bullet}H_{\bullet}(R,\Omega_{\bullet}(M),\Omega(M)) = H^*_{st}(P_x(M))$.

Proof. We refer to [16] or [22, Theorem II.8] for the details of the proof. It is based upon a contraction from $\Omega_{\bullet}(M)^{\otimes n}\otimes\Omega(M)$ into $\Omega_{\bullet}(M)^{n+1}\otimes\Omega(M)$ which to $\omega_1\otimes\cdots\otimes\omega_{n+1}$ associates $\omega_1\otimes\cdots\otimes\omega_{n+1}\otimes 1$ if the degree of ω_{n+1} is strictly larger than 0, and which associates 0 otherwise. ■

Let O be an open subset of M: $P_O(M)$ denotes the set of based paths $\gamma_{\bullet}$ such that $\gamma_0 = x$ and such that $\gamma_1\in O$. We can give the definition of a stochastic plot with values in $P_O(M)$. In Definition 3, we replace the condition iv) by the condition that $\phi_i(u,Z_i(\omega)(u))(\omega)_{\bullet}$ belongs almost surely to $P_O(M)$ if $\omega\in\Omega_i$. Associated to this system of stochastic plots with values in $P_O(M)$, we can define a system of stochastic forms $\Lambda_{st}(P_O(M))$ over $P_O(M)$. There is another way, which was used in [26] to define a stochastic form over $P_O(M)$, by starting from the full set of stochastic plots over the total based path space $P_x(M)$.

Definition 4 *A stochastic form* $\sigma_{st}\in\Lambda'_{st}(P_O(M))$ *is given by the following data: Let* $\psi = (U,\phi_i,Z_i,\Omega_i)$ *be a stochastic plot over* $P_x(M)$. *To* ψ *we associate* $U(\omega)$ *the random open subset of* U *such that, if* $\omega\in\Omega_i$, $\phi_i(u,Z_i(\omega)(u))(\omega)_{\bullet}$ *belongs to* $P_O(M)$. *We associate a random smooth form* $\psi^*\sigma_{st}$ *over* $U(\omega)$, *which checks the analoguous consistency relations ii) and iii) of Definition 2.*

Remark In Definition 1, the parameter set U is deterministic, while in Definition 2, the parameter set $U(\omega)\subseteq U$ is random.

Proposition 1 $\Lambda'_{st}(P_O(M))$ *is included in* $\Lambda_{st}(P_O(M))$.

Proof. The statement is clear, because a stochastic plot with values in $P_O(M)$ is still a stochastic plot with values in $P_x(M)$. Let us show what is the difficulty in order to show the converse inclusion. Let σ_{st} be an element of $\Lambda_{st}(P_O(M))$, and let $\psi = (U, \phi_i, Z_i(u), \Omega_i)$ be a stochastic plot over $P_x(M)$. R^n has a countable basis of neighborhoods. Let U_m, $m \in Z$ be such a basis. We construct a stochastic plot $\psi_m = (U_m, \varphi_i, Z_i, \Omega'_i)$ with values in $P_O(M)$: We consider the subset of Ω_i, denoted by Ω'_i, such that if $u \in U_m$, $\phi_i(u, Z_i(\omega)(u))(\omega)_{\bullet}$ belongs to $P_O(M)$. If ω does not belong to the union of the Ω'_i, we choose any deterministic curve with bounded variation belonging to $P_O(M)$. We deduce a random form over U_m. The problem now is that $U(\omega)$ is an union of U_m and that the set of subsets of Z is not countable. ■

The main property of the set $\Lambda'_{st}(P_O(M))$ is described as follows.

Proposition 2 *Let O_i be a finite open cover of M and g_i be a partition of unity subordinated to it. Let σ^i_{st} belong to $\Lambda'_{st}(P_{O_i}(M))$. Then $\sum g_i(\gamma_1)\sigma^i_{st}$ defines an element of $\Lambda_{st}(P_x(M))$.*

Proof. It is clear, because if we consider a plot $\psi = (U, \phi_i, Z_i, \Omega_i)$ with values in $P_{O_i}(M)$ over $U(\cdot)$, then $\psi^* g_i(\gamma_1)\sigma^i_{st}$ determines a smooth form over U. ■

We will slightly modify the notion of a stochastic form over $L_x(M)$: We will generalize it to the space of paths joining x to y, denoted by $L_{x,y}(M)$.

Definition 5 *A stochastic form σ_{st} over $L_{x,y}(M)$ is given by the following data:*

To each stochastic plot $\psi = (U, \phi_i, Z_i(u), \Omega_i)$ over $P_x(M)$, we associate almost surely a random smooth form $\psi^*\sigma_{st}$ for almost all ω belonging to each Ω_i over the interior $U(\omega)$ of the subset of U such that $\phi_i(u, Z_i(\omega)(u))(\omega_{\bullet})_{\bullet} \in L_{x,y}(M)$.

Moreover, the set of random finite-dimensional forms $\psi^*\sigma_{st}$ satisfy over $U(\omega)$ the analoguous consistency relations ii) and iii) of Definition 3.

The set of stochastic forms in this sense is denoted by $\Lambda'_{st}(L_{x,y}(M))$.

Remark In the Definition 2 ii), the parameter set U is deterministic. In Definition 5, it is random.

We can define the stochastic exterior derivative of a stochastic form in the sense of Definition 5 as in (10). We get stochastic cohomology groups denoted by $H^*(\Lambda'_{st}(L_{x,y}(M))$.

Proposition 3 *For any $y \in M$, $H^*(\Lambda'_{st}(L_{x,y}(M))$ is equal to $H^*(\Lambda'_{st}(L_x(M))$.*

Proof. Let $g_t(z)$ be a smooth family of diffeomorphisms of M extended to a smooth family in $t \in [0,1]$ of diffeomorphisms of $R^{d'}$ with bounded derivatives of all orders as well as their inverse. We suppose that $g_0(x) = x$ and $g_1(y) = x$. Let $\psi = (U, \phi_i, Z_i(u), \Omega_i)$ be a stochastic plot over $P_x(M)$ with values in $L_{x,y}(M)$ over $U(\cdot)$. We associate to it the stochastic plot $g_{\bullet}\psi = (U, g_{\bullet} \circ \psi, Z_i(u), \Omega_i)$. This gives a bijection between stochastic plots with values in $L_{x,y}(M)$ over $U(\cdot)$ and stochastic plots with values in $L_x(M)$ over $U(\cdot)$. Let σ_{st} be an element of $\Lambda'_{st}(L_x(M))$. We associate to $g^*\sigma_{st}$ an element of $\Lambda'_{st}(L_{x,y}(M))$ defined as follows: Let ψ be a stochastic plot with values in $L_{x,y}(M)$ over $U(\cdot)$. We have $\psi^*(g^*\sigma_{st}) = (g\psi)^*\sigma_{st}$. This definition has a sense because $g\psi$ is a stochastic plot with values in $L_x(M)$ over $U(\cdot)$. Clearly, g^* induces a bijection from $\Lambda'(L_x(M))$ to $\Lambda'_{st}(L_{x,y}(M))$ which is compatible with the stochastic exterior derivative because:

$$\psi^*(d(g^*\sigma_{st})) = d\psi^*(g^*\sigma_{st}) = d(g\psi)^*\sigma_{st} = (g\psi)^*(d\sigma_{st}) = \psi^*(g^*d\sigma_{st}). \quad (19)$$

Therefore the proposition holds. ■

Remark This isomorphism property is uniquely based upon the Itô formula and is much more simpler to prove than the isomorphism formula for stochastic cohomology groups in the sense of Nualart-Pardoux [23, 24] based upon the Albeverio-Hoegh-Krohn quasi-invariance formulas or upon Driver's flow, which brings into heavy technicalities. This remark also relevant for the following proposition.

Proposition 4 *Let O be a small convex neighborhood of y in M. Then the group $H^*(\Lambda'_{st}(L_{x,y}(M)))$ is equal to $H^*(\Lambda'_{st}(P_O(M)))$.*

Proof. Let $\lambda \in [0,1]$ and $t \in [0,1]$. We consider a smooth family of maps parameterized by λ and t from $R^{d'} \times R^{d'}$ into $R^{d'}$ with bounded derivatives of all orders such that:

1) $g_{\lambda,0}(x,z') = x$.

2) $z \to g_{\lambda,t}(z,z')$ is a diffeomorphism of M with bounded derivatives of all orders as well as its inverse. This diffeomorphism is equal to Id_M at $t = 0$.

3) $z \to g_{\lambda,t}(z,y)$ is equal to Id_M.

4) $z \to g_{1,t}(z,z')$ is equal to Id_M.

5) $z \to g_{\cdot,1}(z,z)$ realises a retraction from O onto y.

Let $\psi = (U, \phi_i, Z_i(u), \Omega_i)$ be a plot in $P_x(M)$ and $s \to g_{\lambda,s}(\psi_s, \psi_1)$ be its associated plot. We get another plot in $P_x(M)$, denoted $g_{\lambda,\bullet}(\psi)$. If ψ is in $P_O(M)$ over $U(\cdot) \subseteq U$, $g_{\lambda,\bullet}(\psi)$ is still in $P_O(M)$ over the same $U(\cdot)$, if $\lambda \neq 0$. We deduce a map $g^*_{\lambda,\bullet}$ from $\Lambda'_{st}(P_O(M))$ into $\Lambda'_{st}(P_O(M))$. If $\lambda = 0, g_{0,\bullet}(\psi)$ is a plot with values in $L_{x,y}(M)$ over $U(\cdot)$. We deduce a map $g^*_{0,\bullet}$ from $\Lambda'_{st}(L_{x,y}(M))$ into $\Lambda'_{st}(P_O(M))$ as in the proof of Proposition 3, which preserves the exterior stochastic differential. We deduce a map $g^*_{0,\bullet}$ from $H^*(\Lambda'_{st}(L_{x,y}(M))$ into $H^*(\Lambda'_{st}(P_O(M))$. Let i^* the inclusion map from $\Lambda'_{st}(P_O(M))$ into $\Lambda'_{st}(L_{x,y}(M))$, which has a sense by the definition of

$\Lambda'_{st}(L_{x,y}(M))$. This extends to a map of the cohomology group $H^*(\Lambda'_{st}(P_O(M)))$ into $H^*(\Lambda'_{st}(L_{x,y}(M)))$.

By 3), $i^* g^*_{0,\bullet} = I_d$ in cohomology, because it is the identity at the level of forms.

Let us show that $g^*_{0,\bullet} i^* = Id$ in cohomology. The proof closely follows the proof of Lemma II.7 of [26], Lemma 4.9 of [28], Theorem III.3 of [27], and Theorem 1. We consider the plot ψ^{ext} indexed by $U \times [0,1]$, $g_{\lambda,\bullet}(\psi)$. Suppose that ψ takes its values in $P_O(M)$ over $U(\cdot) \subseteq U$. The main remark is the following: ψ^{ext} still takes its values in $P_O(M)$ over $U(\cdot) \times [0,1]$. Suppose that σ_{st} over $P_O(M)$ is closed. We deduce a random finite-dimensional form over $U(\cdot) \times [0,1]$ $(\psi^{ext})^*\sigma_{st}$. We have a retract from $U \times [0,1]$ into U: $(u,\lambda) \to (u, f_\varepsilon(\lambda))$. We apply the Cartan formula as in (2.16) of [26] or (14) of Theorem 3, λ and ε being interchanged. By (2.17) of [26], we deduce that $\sigma_{st} = g^*_{0,\bullet} i^* \sigma_{st} + d\tilde{\sigma}_{st}$. This shows the result. ■

We can now state the main result of this section.

Theorem 3 *Let us suppose that M is simply connected. Then the cohomology group $H^*(\Lambda'_{st}(L_x(M)))$ is equal to $H_\bullet H_\bullet(R, \Omega_\bullet(M), R)$.*

Proof. Let O_i be a cover by small open convex subsets of M. Let $O_{i_0,\dots,i_r} = \cap O_{i_j}$. We consider the set $\Lambda'_{st}(P_{O_{i_0,\dots,i_r}}(M)$ endowed with 2 gradings:

1) r, the length of the intersection minus 1.

2) The degree of the stochastic form.

There is a bicomplex associated to this bigrading. It is the Cech complex δ in the length of the intersection, and, in the degree of the stochastic form, it is the stochastic exterior derivative.

There are two spectral sequences which converge to the total cohomology of the bicomplex. But by Proposition 2, $\tau_{st} = \sum g_i(\gamma_1)\sigma_{st,i,i_0,\dots,i_{r-1}}$, where g_i is a smooth partition of unity associated to O_i, is a stochastic form such that $\delta\tau_{st} = \sigma_{st}$. Therefore the two spectral sequences converge to the stochastic cohomology groups of the path space.

The remaining of the proof is exactly the same as the remaining of the proof of this statement for Sobolev cohomology groups of the loop space (see [23, 24, 16]).

We consider a small contractible open subset of M: We see that the cohomology group $H_\bullet H_{\bullet O}(R, \Omega_\bullet(M), \Omega^O(M))$ is equal to the cohomology group $H_\bullet H_\bullet(R, \Omega_\bullet(M), R)$ where in $H_\bullet H_{\bullet O}(R, \Omega_\bullet(M), \Omega^O(M))$ we consider the last form ω_n in (18) which is a form over O instead of M. (See [23, Lemma II.6].) Over the set of Chen forms with support in $O_{i_0,\dots,i_r}$ included in M, there is a bigrading: The length of the intersection $O_{i_0,\dots,i_r}$ and the degree associated to the Hochschild boundary which corresponds to the degree of the associated form over the path space. We get a spectral sequence where the Hochschild boundary (or the stochastic exterior derivative) is the vertical complex and the horizontal complex is the

Cech complex. The map stochastic Chen iterated integral preserves these 2 spectral sequences.

The first term $E_1^{p,q}$ of the spectral sequence is the space of the Cech complex with values in the presheaf $H^q(\Lambda'_{st}(P_O(M)))$ which is locally constant because $H^*(\Lambda'_{st}(P_O(M)))$ is equal to $H^*(\Lambda'_{st}(L_x(M)))$ for a small open contractible set O. But it is constant since M is simply connected. Therefore $E_2^{p,q} = H^p(M) \otimes H^q(\Lambda'_{st}(L_x(M))$.

It is the same for the spectral sequence associated to the Hochschild complex. Therefore $E_2^{p,q} = H^p(M) \otimes H_{\bullet}H_{\bullet}^q(R,\Omega_{\bullet}(M),R)$.

The map iterated stochastic Chen integral induces a map of spectral sequences. It induces an isomorphism by Theorem 3 on E_∞. If it induces an isomorphism on $E_2^{p,0}$, it will induce by the Zeeman comparison theorem [32] an isomorphism on $E_2^{0,q}$. These facts will show the theorem.

But $H^0(\Lambda'_{st}(L_x(M))) = R$. Namely, $H^0(\Lambda_{st}(P_x(M))) = R$ (see Theorem 2). We consider the spectral sequence associated to the Cech coboundary δ as horizontal complex and to the stochastic exterior derivative d as vertical complex. $E_2^{p,q} = H^p(M) \otimes H^q(\Lambda'_{st}(L_x(M)))$, because M is simply connected. Since we have a first quadrant spectral sequence, then $E_l^{0,0} = E_\infty^{0,0}$ and $E_\infty^{0,0} = H_{st}^0(P_x(M)) = R$. (See [4] for a proof in the cae of smooth loop space.) The same argument holds for $H_{\bullet}H_{\bullet}^0(R,\Omega_{\bullet}(M),R)$. ■

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Are there macroscopic quantum bodies?

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Abstract. The validity of quantum laws in the macroscopic scale is discussed. Some size and time distortions predicted by quantum control theory can imitate the effects of the $\hbar$-deformation, helping us to observe the quantum behavior of huge micro-objects.

Keywords: Schrödinger's paradox, quantum control, time distortion

Mathematics Subject Classifications (2000): 81P99, 81Q05

> La physique est en effet une science qui n'avance que lorsqu'elle rencontre un paradoxe...
>
> Moshé Flato
> *Le pouvoir des mathématiques*
> *(Hachette 1990, p. 44)*

1. Introduction

One of puzzles of the present day physical theory is the fundamental difference between the classical and quantum structures. While the classical observables are just c-number functions on the classical phase space, the quantum ones form non-commutative operator algebras [1, 2, 3, 4]. The difference is especially critical on the conceptual level, in problems concerning measurement theory. It is enough to remember that one of the crucial points in the Copenhagen interpretation was the fundamental distinction between the micro-objects and the macroscopic measuring devices [5, 6, 2, 7], an idea which has raised multiple discussions. Indeed, should the measuring devices obey certain exceptional laws just because they were created to measure something? What are they if not massive bodies (agglomerations of micro-objects) sensitive to the presence of quanta? Should therefore the laws of quantum mechanics be not applicable to certain macroscopic bodies?

The orthodox interpretations (including the Copenhagen school [5, 2, 6]) try to disarm the paradox by arguments similar to the "double thinking" of George Orwell [8]. On one hand, they insist that (A) the quantum theory applies to arbitrarily huge physical systems (including the entire universe). On the other hand, they tell you, that (B) you should not apply it to the measuring devices, which are essentially classical. Thus, when trying to understand the measurement process, we should forget completely that the measuring device is a huge quantum system (or at

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least, we must keep this thought in the back of our minds). The motive persistently returns: When remembering (A) you must forget (B), and when remembering (B) you must forget (A).

The interpretational strategy would be almost successful were it not for the opposite trend which tries to exhibit the depth of the problem. Indeed, by treating seriously the linear evolution laws for quite arbitrary systems, one unavoidably ends up with macroscopic bodies which should be in the superposed states corresponding to different macroscopic effects. Thus, there should exist an undecided macroscopic observer in a superposed state of 'perceiving two different measurement results' on his macroscopic apparatus (and moreover, contemplating them both in his superposed brain). The most dramatic example was given by Erwin Schrödinger [9] in 1935, by considering an animal (a cat) in a superposed state of being alive and being dead (Schrödinger, like Einstein, opposed the philosophy of the Copenhagen school [7, 10, 11]). Yet, should such states be forbidden? Are they the physical reality or an exaggeration of the doctrine? This last possibility is considered by Ghirardi, Rimini, Grassi, Diossi [15, 16, 17], and Roger Penrose [18, 19], who believe that the quantum linearity should break for heavy objects.

The problem, in a sense, can serve to classify the schools of thinking. Some of them are aware of the creative role of the paradox, and henceforth, enjoy the difficulty, even if provoking *un vent de panique dans la corporation* (M. Flato [20], p. 44). The others try to evade the question (arguing that the physicist should think about graphs, not about metaphysical puzzles!)

Thanks to new empirical techniques, however, the puzzle is again in the center of discussion.

In a sequence of papers on the "Schrödinger's cat states" the authors try to construct the mesoscopic states sharing the situation of the Schrödinger cat. Until now, no size or mass limits are noticed; the unique challenge appears to be the decoherence phenomenon [21, 22]. An independent search in the neutron interferometry has proved the reality of the quantum superposition for the neutrons with the wave packet components separated by distances of the order of magnitude of centimeters [24]. As interesting are the effects of collective tunneling on the Josephson junction (see, e.g. Caldeira and Legget [25, 26]). A remarkable event of the nineties was the experimental observation of the interference pictures for sodium and beryllium atoms [27]. In the subsequent interference experiments with beams of Na_2 molecules, the authors could report: "here we coherently manipulate exclusively the center-of-mass motion" [28]. More striking even were the analogous diffraction experiments for large (about 6nm), weakly bound, fragile He_2 molecules [29] which, unless they break when passing through the gratings, show the interference patterns corresponding to the de Broglie waves of their mass centers. While the interference of atoms is already explored to construct gradiometers [30], all previous results were left behind by the observations of interference patterns for the soccer ball-shaped molecules C_{60}, passing with velocities about 200m/sec through a system of slits 50nm wide, 100nm apart [31].

In all these experiments the Planck constant is fixed. Yet, the effects are curiously related to the subject of deformation [32, 33, 34, 35, 36, 37]. Its main point is a "comparative anatomy": Our universe is best understood if submerged in an ample class of *virtual universes* where the physical laws could be altered and physical constants changed. The same motive is present in relativistic cosmology (our universe is one, but cosmological models are many!). The idea has also found some artistic expressions. In their excellent novel *The Compleat Enchanter*, L. Sprague de Camp and F. Pratt illustrate the meaning of our physical laws by letting their heroes visit a parallel universe where these laws are broken [38]. In the well known S/F story by George Gamow [39], an expedition of adventurers travels to a distant jungle where the Planck constant is greater than in the rest of the universe. What they face is a deformed version of our reality: The gazelles form interference patterns after running through the bamboo wood. The old elephants have their skins visibly fuzzy: The result of a slow dissolution of the wave packets. The entire story is naive (the single elephant does not need to look fuzzy, even if its mass center is uncertain.) Yet, can it tell us something?

As it seems, Gamow had some right intuitions, which can be formulated much better now [33, 34]. The $\hbar$-deformation is a natural paradigm to see the true magnitude of the Schrödinger problem (the *caricature* helps to see the reality!) Thus, e.g., consider a planetoid in cosmic space. When thinking about a cosmic body it is difficult to avoid a certain oversimplified picture: It includes the intrigues of the star-trek, but it does not include the Planck constant. Yet, even if small, this constant exists. If the quantum theory is universal, the mass center of the planetoid after a sufficiently long time should show some quantum uncertainty. The effect is extremely small, but it can be of fundamental importance. (As the matter of fact, the quantum diffusion of heavy blocks of metal was carefully taken into account as an obstacle to detect the gravitational waves [40]). Imagine now, that some beings live on the surface of our planetoid, and can observe the surrounding universe. What will they see? Since the planetoid center of mass is diffused, will the position of the distant stars be "fuzzy" in their eyes? Is the problem limited to a planetoid? Could one quantum object become a reference frame for other quantum objects?

The idea of *quantum frames* was put forward in 1984 by Aharonov and Kaufherr [41]; in a different form by Wódkiewicz [42]. Apart of its philosophical appeal, the problem cannot be avoided in several areas, including quantization of gravity. Yet, perhaps due to our macroscopic clumsiness, the idea has merely emerged... In a world with a huge Planck constant, it would already cause a 'wind of panic' (the space is crossed by *quantum planetoids*, the fundamental concept of the *inertial frame* is affected by the quantum deformation!). In our own universe, though, $\hbar$ is so small and the effects so tiny, that the creative panic still did not arrive...

While we are still waiting, let me report some little steps of control theory which can compensate the smallness of $\hbar$, bringing us closer to the moment of the confrontation.

2. Size deformations

Even if quantum mechanics holds for macroscopic objects, their quantum behavior (interference) may be masked by a number of effects making increasingly difficult the repetitions of Young's experiment:

1) de Broglie waves of heavy micro-objects are very short;

2) For heavy microscopic objects the wave fronts are narrow and diffract slowly; the wave packets resemble needle beams propagating along the classical trajectories [43].

A way to avoid the first difficulty could be to create beams of extremely slow microparticles (for charged particles this can be achieved by applying electric fields; for neutral ones, the Doppler cooling is available [44, 45]).

To avoid the second difficulty one can try to split the wave packet on an obstacle, but above C_{60} the method might fail (the packet will be split but the coherence might be lost). Yet, for charged particles, the quantum control theory admits a class of operations which can widen the wave packets without destroying the coherence.

The phenomena of *squeezing/amplification* were originally discovered for the photon states in quantum optics [46]. Soon, it was shown that they can be repeated for charged massive particles. In the 1-dimensional non-relativistic quantum mechanics the simplest such effect is produced by the scale operator:

$$U = e^{i\mu\frac{qp+pq}{2}}, \qquad \mu \in \mathbf{R}, \tag{1}$$

transforming the Schrödinger wave packets according to:

$$\psi(x) \to (U\psi)(x) = \frac{1}{\sqrt{\sigma}}\psi\left(\frac{x}{\sigma}\right), \qquad \sigma = e^{\mu}. \tag{2}$$

An elementary technique to generate (1–2) as an evolution operation consists in applying time-dependent oscillator potentials; the Hamiltonian $H(t)$ and the evolution operator $U(t, t_0)$ are given by:

$$H(t) = \frac{p^2}{2} + \gamma(t)\frac{q^2}{2} \tag{3}$$

$$\frac{dU}{dt} = -iH(t)U(t,t_0) \tag{4}$$

(we put $m = \hbar = 1$). Note, that (1–2) can be easily generated inside of the ion traps [47]. The operational programs for $\gamma(t)$ to produce (1–2) at an appointed time $t = \tau$ apply multiple methods, [46, 48, 49, 50, 54, 51, 52, 53], helped by the formalism of Milne (1926) [55] and by elements of the Sturm-Liouville theory. As a matter of fact, we shall show that one of the traditional Sturm-Liouville problems, when carefully read, gives all necessary information about the squeezing together with the numerical data needed to program the phenomenon. The simplest key is contained in the following *Helmholtz equation* [56]:

$$\frac{d^2\xi}{dx^2}+\lambda\phi(x)\xi(x)=0. \tag{5}$$

The *eigenvalue problem* of Helmholtz consists in looking for the λ-values which permit non-trivial solutions of (5) vanishing on the extremes of a certain fixed interval $[a,b]$:

$$\xi(a)=\xi(b)=0. \tag{6}$$

To see the meaning of (5–6) for the generation of the *squeezed states* it is enough to apply a sequence of simple reinterpretation steps.

Step one. We interpret (5) as a classical motion problem for a point particle on the Poincaré plane. Denote: $x=t$ (and call it "time"); $\xi=q$, $\xi'=p$; then (5) leads to the pair of canonical equations:

$$\begin{aligned}\frac{dq}{dt} &= p\\ \frac{dp}{dt} &= -\lambda\phi(t)q\end{aligned} \tag{7}$$

which describe a classical point particle obeying the Hamiltonian (3) with $\gamma(t)=\lambda\phi(t)$. Now, if $\phi(t)\geq 0$, $\phi\neq 0$ and $\lambda>0$, the attractive oscillator in (7) transports the classical point $(q(t),p(t))$ around the phase space origin. Let the initial point be $q(a)=0$, $p(a)\neq 0$, then as λ tends to infinity, the end point $q(b)$, $p(b)$ must cross the p-axis an increasing number of times. Each time this happens, λ is an eigenvalue of (1–2). Thus, the eigenproblem (5–6) has a discrete spectrum forming an infinite ladder of positive eigenvalues. However, the spectral problem (5–6) still contains more information.

Step two. We read the optical data. In the typical Sturm Liouville problem one looks only for the solution of (5) which fulfills the boundary condition (6). The second independent solution is forgotten as unphysical. Yet, this solution precisely brings an interesting message. Since (7) are linear and homogeneous, the solution $q(t)$, $p(t)$ for any λ depends linearly on the initial values:

$$\begin{pmatrix} q(t)\\ p(t)\end{pmatrix}=u(t,a)\begin{pmatrix} q(a)\\ p(a)\end{pmatrix}, \tag{8}$$

where $u(t,a)$ is the symplectic *evolution matrix* for the time interval $[a,t]$. Consider now the matrix $u(b,a)$. Note that it must have a special form whenever λ is one of the spectral values of (5–6). Indeed, the existence of a non-trivial $q(t)$ with $q(a)=q(b)=0$ means that $u(b,a)$ transforms $(0,\,p(a))$ into $(0,\,p(b))$; henceforth, it must be of the form:

$$u(b,a)=\begin{pmatrix}\sigma & 0\\ \eta & 1/\sigma\end{pmatrix}, \qquad \sigma,\eta\in\mathbf{R},\ \sigma\neq 0. \tag{9}$$

The matrix (9) gives now an information about the trajectories of (5) *which do not fulfill the boundary condition* (6). In fact, consider the congruence of classical orbits which depart at $t = a$ from a certain common initial point (q_0, p), (q_o fixed $\neq 0$, p varies). If λ is an eigenvalue of (5–6), then $u(b,a)$ is of form (9), and (8) implies:

$$\begin{pmatrix} q(b) \\ p(b) \end{pmatrix} = \begin{pmatrix} \sigma q_0 \\ \eta q_0 + \frac{p}{\sigma} \end{pmatrix}. \tag{10}$$

By plotting $q(t)$ against t, one sees that all orbits which diverge from a common point q_0 with varying p's at $t = a$ must meet for $t = b$ at a common point σq_0 with varying $p' = \eta q_0 + p/\sigma$. In optical terms, σq_0 is *an image* of q_0 (squeezed if $|\sigma| < 1$, natural if $|\sigma| = 1$, amplified if $|\sigma| > 1$). The eigenvalues λ of (5–6), are *the only real numbers* for which the congruencies obeying (7), departing from any common q_0 at $t = a$ must focus in a common point σq_0 at $t = b$. The optical constant σ is the squeezing (amplification) constant for the disregarded solutions of (5–6) [57].

Step three. We notice that the optical effect is not limited to the classical orbits. In fact, for the quadratic Hamiltonians, Equations (7) remain valid for the time-dependent canonical operators $q(t)$, $p(t)$, and the matrix (9) represents the dynamical evolution of the position and momentum observables in the Heisenberg frame. Thus, if $U(t,a)$ is the unitary evolution operation in $[a,t]$, if $q(a) = q$ and $p(a) = p$ are the initial canonical operators, and if $\lambda = \lambda_n$ is one of the eigenvalues $0 < \lambda_1 < \lambda_2 \ldots$, the evolution result at $t = b$ is:

$$U(b,a)^\dagger\, qU(b,a) = \sigma q \tag{11}$$

$$U(b,a)^\dagger pU(b,a) = \eta q + \sigma^{-1} p \tag{12}$$

(η, σ depending on n). Consistently, the evolution operator is [57]:

$$U(b,a) = P^n e^{i\ln|\sigma| \frac{qp+pq}{2}} e^{i\eta \frac{q^2}{2}} \tag{13}$$

(P is the parity operator) and transforms the wave packets according to:

$$\psi_U = [U(b,a)\psi] = \frac{1}{\sqrt{\sigma}} e^{i\eta \frac{x^2}{2\sigma^2}} \psi\left(\frac{x}{\sigma}\right) \tag{14}$$

producing the scale operation ("size deformation") corrected by an x-dependent phase factor. What is essential, the packets undergo the size deformation without losing coherence [57]. Interestingly, if the function $\phi(t)$ is symmetric in $[a,b]$, then the effect is trivial ($|\sigma| = 1$), but for ϕ asymmetric, in general, $|\sigma| \neq 1$. For $|\sigma| > 1$, the method can be used to amplify the narrow wave packets of heavy micro-objects to exhibit their quantum behavior.

Step four. The method now permits to use the magnetic operations in order to squeeze/amplify the states of charged micro-objects in more than one dimension. Indeed, consider a homogeneous, time-dependent magnetic field $\mathbf{B}(t) = B(t)\mathbf{n}$ of a cylindrical solenoid. The correspondence between the time-dependent magnetic fields and the oscillators of the form (3–4) on the plane orthogonal to the solenoid axis, makes possible to program the magnetic pulses which generate the scale transformation of the charged wave packet in the plane orthogonal to **n** (this can be achieved, in particular, by applying the sinusoidally oscillating magnetic fields $B(t) = B_0 + B_1 \sin\omega t$ of adequate amplitudes and frequency [57, 58]). If the effect is an expansion, then apart of producing the amplified images in terms of classical trajectories, it will also magnify the wave front of each single particle in the solenoid. In theory, maintaining properly oscillating magnetic fields, one can expand the wave fronts of arbitrarily heavy, charged point-like bodies to any desired size, bringing them into a controversial state of macroscopic quantum superpositions.

3. Time distortions

The possibility of analogous *time distortions* is due to the phenomenon of closed evolution processes (evolution loops). An obvious loop is produced by the harmonic oscillator (but brings no consequence for our subject). The possibility of more general loop phenomena was known already in 1970 [59]. The first evolution loop containing intervals of the free evolution (and thus, making possible the time deformations), was accidentally found in 1977 [60] as an exact solution of the Baker-Campbell-Hausdorff problem (BCH) for the product of 12 exponential terms:

$$e^{-i\frac{p^2}{2}} e^{-i\frac{q^2}{2}} \dots e^{-i\frac{p^2}{2}} e^{-i\frac{q^2}{2}} \equiv 1, \tag{15}$$

where $\equiv$ denotes the operator proportionality up to a phase factor ($K \equiv M$ means $K = e^{i\kappa} M$, where $\kappa \in \mathbf{R}$). Physically, the sequence (15) stands for the 6 operations of the free evolution (in time lapses $\tau = 1$) interrupted by 6 kicks of the attractive oscillator potential. The identity (15) is an *echo phenomenon* for the non-spin states; it means that the Schrödinger's particle, affected by the entire sequence of 12 operations, returns to its initial state (whatever this state was).

As subsequently shown, the loop (15) is just a particular case of a large family of loop identities solving exactly the BCH problem for the generators of $SO(1,2)$. Most of them suggest technologies of the "time manipulation". The simplest case is the following exponential identity [61]:

$$e^{-i\gamma\frac{p^2}{2}} e^{-i\alpha\Gamma\frac{q^2}{2}} e^{-i\beta\frac{p^2}{2}} e^{-i\gamma\Gamma\frac{q^2}{2}} e^{-i\alpha\frac{p^2}{2}} e^{-i\beta\Gamma\frac{q^2}{2}} \equiv 1, \qquad \Gamma = \frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}. \tag{16}$$

Taking $\gamma = -\tau$ one obtains the formula for the free evolution achieved in a "distorted time":

$$e^{-i\tau\frac{p^2}{2}} \equiv e^{-i\alpha\Gamma\frac{q^2}{2}} e^{-i\beta\frac{p^2}{2}} e^{+i\tau\Gamma\frac{q^2}{2}} e^{-i\alpha\frac{p^2}{2}} e^{-i\beta\Gamma\frac{q^2}{2}}. \tag{17}$$

If $\tau < 0$, (17) describes the retrospection operation (free evolution inversion) achieved in the laboratory time $T = \alpha + \beta$ at the cost of three sharp pulses of an attractive oscillator potential. If $0 < \tau \leq T$, the sequence (17) produces the free evolution slowing, achieved again in $T = \alpha + \beta$ with the help of two negative (repulsive) and one positive (attractive) pulses. If $\tau > \alpha + \beta$, the free evolution is accelerated by a program of two positive and one negative oscillator pulses. Note that the first case can be immediately translated into a program of homogeneous magnetic pulses in a cylindrical solenoid causing the ‘retrospection’ of the Schrödinger wave function on the plane orthogonal to the solenoid axis (the z-dependence still obeying the ordinary free evolution). However, it is the third case (time acceleration) which might turn pertinent for our problem.

A different method of altering the time was designed in 1990 by Aharonov et al. [62]. It consists in controlled evolution interrupted by a measurement. The existence of the macroscopic superposition in this method is *a priori* assumed and the process is dissipative: A part of the ensemble is lost but a part which survives recovers its pure state from an arbitrarily deep past. As it seems, a variant of the same method could be used as well to accelerate rather than to invert the evolution.

Meanwhile, the schemes of non dissipative (pure) magnetic operations causing the time distortions of all three types were further elaborated in 1994 [63]. One of the solutions consists in a sequence of 12 rectangular pulses of homogeneous magnetic fields of amplitudes

$$B_1, B_2, -B_2, -B_1,\; B_1, B_2, -B_2, -B_1,\; B_1, B_2, -B_2, -B_1$$

arriving in 3 packets, each one composed of 4 parallel pulses $B_1, B_2, -B_2, -B_1$, subsequently in x, y, and z directions. The simplest scenario might involve 3 sources of homogeneous, mutually orthogonal magnetic fields (e.g., three electromagnets) activated subsequently. The time duration of each pulse of intensity B_i is τ_i (no matter the direction); the sum $\tau = 6(\tau_1 + \tau_2)$ is the total operation time. As can be calculated, the output depends on the dimensionless parameters $\beta_i = eB_iT/2mc$, $\vartheta_i = \tau_i/T$, where $T = \tau_1 + \tau_2$ is the time duration of each pulse pair B_1, B_2. Only if

$$\beta_1 \tan(\vartheta_1\beta_1) + \beta_2 \tan(\vartheta_2\beta_2) = 0 \tag{18}$$

the dynamical operation upon the charged particle in $\mathbf{R}^3$ generated by the 12 rectangular magnetic pulses is of the free evolution type:

$$U(\tau, 0) \equiv e^{-i\tau' \frac{\mathbf{p}^2}{2}}, \tag{19}$$

but the parameter τ' does not coincide with the real operation time τ. One has:

$$\tau' = \theta\tau, \tag{20}$$

where θ is the *deformation coefficient* [63]:

$$\theta = \frac{1}{3} + \frac{2}{3}\cos^2\gamma_2 \frac{\tan^2\gamma_2}{\gamma_1\tan\gamma_1 - \gamma_2\tan\gamma_2}, \quad \gamma_i = \beta_i\vartheta_i. \tag{21}$$

Similarly as before, our formula was obtained for a sequence of Hamiltonians neglecting the field retardation (an approximation which might be valid at the laboratory scale). The algebraic identities are elementary (an exact case of BCH) though not without a physical implication. When repeated persistently, the operations can deform in a coherent way the free evolution of a heavy charged molecule, so that it will become a laboratory equivalent of one of George Gamow elephants. If such objects indeed exist.

It is my pleasure to dedicate these thoughts to the memory of Moshé Flato, an exceptional personality, who can hardly be imitated in the present day mathematical physics.

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Star-products, spectral analysis, and hyperfunctions

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Abstract. We study the $\star$-*exponential* function $\mathscr{U}(t;X)$ of any element X in the affine symplectic Lie algebra of the Moyal $\star$-product on the symplectic manifold $(\mathbb{R}\times\mathbb{R};\omega)$. When X is a compact element, a natural specific candidate for $\mathscr{U}(t;X)$ to be the exponential function is suggested by the study we make in the non-compact case. $\mathscr{U}(t;X)$ has singularities in the t variable. The analytic continuation $\mathscr{U}(z;X), z=t+iy$, defines two boundary values $\delta_+\mathscr{U}(t;X)=\lim_{y\downarrow 0}\mathscr{U}(z;X)$ and $\delta_-\mathscr{U}(t;X)=\lim_{y\uparrow 0}\mathscr{U}(z;X)$. $\delta_+\mathscr{U}(t;X)$ is a *distribution* while $\delta_-\mathscr{U}(t;X)$ is a Beurling-type, Gevrey-class $s=2$ *ultradistribution*. We compute the Fourier transforms in t of $\delta_\pm\mathscr{U}(t;X)$. Both Fourier spectra are discrete but different (e.g. opposite in sign for the harmonic oscillator). The Fourier spectrum of $\delta_+\mathscr{U}(t;X)$ coincides with the spectrum of the selfadjoint operator in the Hilbert space $L^2(\mathbb{R})$ whose Weyl symbol is X. Only the boundary value $\delta_+\mathscr{U}(t;X)$ should be considered as the $\star$ -exponential function for the element X, since $\delta_-\mathscr{U}(t;X)$ has no interpretation in the Hilbert space $L^2(\mathbb{R})$.

Keywords: deformation of algebras, hyperfunction, quantization

Mathematics Subject Classifications (2000): 53D55, 81S30, 58G07

Soon after the founding papers [4, 5] appeared, Moshé Flato suggested that further study was needed for the $\star$-*exponential* and its Fourier transform, for functions defined on a symplectic manifold endowed with a $\star$-product. This led us in 1980 to the computations presented here [18, 19]. In particular, the idea of looking at the $\star$-exponential as a boundary value of some holomorphic function, that is to say as a hyperfunction, was also suggested to us by Moshé.

We dedicate this paper to Moshé with deep gratitude and affection.

1. Introduction

We consider the canonical symplectic manifold $(\mathbb{R}\times\mathbb{R};\omega)$, the Moyal $\star$-product on this manifold, and the affine symplectic Lie algebra $\mathscr{A}$ of Hamiltonians corresponding to the infinitesimal automorphisms of the Moyal star-product. This Lie algebra is the vector space of polynomials of degree ≤ 2 in the coordinates p and q endowed with the Lie algebra structure induced by the Poisson bracket defined by ω.

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The aim of this paper is to study the $\star$-*exponential* of any element of $\mathscr{A}$:

$$\mathscr{U}(t;X) = \sum_{0}^{\infty} \frac{1}{n!}(it)^n (X*)^n, \tag{1}$$

and in particular its Fourier transform in t.

In Section 2 we introduce the Moyal $\star$-product. Since $X \in \mathscr{A}$, we may expect the function (1) to be of the type $f(t;X(p;q))$. Indeed the operator $X*$ acting on functions of this type becomes a second-order differential operator in the X variable. This is proved in Lemma 2. We also specify the two cases, that will be separately considered: non-compact elements of $\mathscr{A}$, corresponding to the notation $d<0$ in this paper, and compact elements corresponding to $d>0$.

In Sections 3 and 4, when $d<0$, we solve the eigenvalue problem [5] :

$$X \star \pi(\lambda;X) = \lambda \pi(\lambda;X), \quad \lambda \in \mathbb{R}. \tag{2}$$

We see that this equation is nothing but the Kummer equation [23] for which the hypergeometric confluent function ${}_1F_1[a;b;X]$ is a solution. The study of this equation leads us in Section 4 to the following expression for the function (1):

$$\mathscr{U}(t;X) = \left(\cosh \frac{t}{|s|}\right)^{-1} e^{i(|s|(X+c)\tanh(\frac{t}{|s|})-ct)}. \tag{3}$$

Using an integral representation for the Kummer function, similar to those found in [23] and given in Section 3, it is possible to compute the Fourier transform in t of (3): It is just $\frac{1}{2\pi}\pi(\lambda;X)$. From the point of view of the corresponding Weyl quantization, this exponential function is the Weyl symbol of a one-parameter group of unitary operators on the space $L^2(\mathbb{R})$ generated by the selfadjoint operator whose Weyl symbol is $X(p\,;q)$ [9, 14, 18, 19]. The above Fourier transform of (3) determines the spectral resolution of that one-parameter group of unitary operators.

In Section 5 we handle the case $d>0$ in a similar way. We begin with the Kummer equation (14) and transform it by substituting formally $X \to iX$; $t \to it$; $c \to ic$; $i \to -i$. We then arrive at the formal result that the exponential function for this case would be [5, 6, 14]

$$\mathscr{U}(t;X) = \left(\cos \frac{t}{|s|}\right)^{-1} e^{i(|s|(X+c)\tan(\frac{t}{|s|})-ct)}. \tag{4}$$

The exponential function cannot be completely defined $\forall(t;X) \in \mathbb{R}\times\mathbb{C}$ by our procedure because of singularities appearing in the t variable of (4). It is therefore natural [18, 19][3] to take the function (4) as a boundary value [10, 15, 17] of the holomorphic function

$$\mathscr{U}(z;X) = \left(\cos \frac{z}{|s|}\right)^{-1} e^{i(|s|(X+c)\tan(\frac{z}{|s|})-cz)}, \quad \text{where } z = t+iy. \tag{5}$$

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[3] It has been pointed out to the authors that the idea of taking the boundary values of functions in the convergent $\star$-product problem was used independently in a recent paper [20].

In Section 6 we prove that for $y \to 0$, $y > 0$, the boundary value of (5) is a distribution that we denote by $\delta_+\mathscr{U}(t;X)$; and for $y \to 0$, $y < 0$, the boundary value is an ultradistribution that we denote by $\delta_-\mathscr{U}(t;X)$, the type of which we determine in this Section 6. We also prove some properties of $\delta_\pm\mathscr{U}(t;X)$ in Proposition 7.

The computation of the Fourier transform in t of the hyperfunctions $\delta_\pm\mathscr{U}(t;X)$ is presented in Section 7. The Fourier spectrum is discrete in both cases. For $\delta_+\mathscr{U}(t;X)$, we get [9] what corresponds to the Weyl quantization of a selfadjoint operator on the space $L^2(\mathbb{R})$, whose Weyl symbol is $X(p;q)$, and $\delta_+\mathscr{U}(t;X)$ is the Weyl symbol of the one-parameter group of unitary operators generated by the same selfadjoint operator [5, 14, 18, 19]. Furthermore, the expressions (34) and (32) determine the Weyl symbol of the spectral resolution of this operator, the spectrum of which is (37): $\lambda_{N_0} = \frac{2}{|s|}(N_0 + \frac{1}{2}) - c$, $N_0 \in \mathbb{N}$. There is no Hilbert-space interpretation in $L^2(\mathbb{R})$ for $\delta_-\mathscr{U}(t;X)$. Nevertheless [18, 19] we can obtain Fourier expansions similar to those for $\delta_+\mathscr{U}(t;X)$, such as (35) and (33) which give the spectrum (38): $\lambda_{N_0} = -\frac{2}{|s|}(N_0 + \frac{1}{2}) - c$, $N_0 \in \mathbb{N}$. In particular, for the harmonic oscillator, these spectral values are *precisely the opposite* of the correct ones obtained with $\delta_+\mathscr{U}(t;X)$.

As we see in this paper, in order to obtain the set of spectral values of the classical observable $X(p;q)$ in $\star$-quantization, we must complexify time, $t + iy$, and then choose one of the above boundary values. At this point it is worthwhile mentioning that complexifying t appears also in a formulation of both classical and quantum-statistical mechanics in terms of star-products, [2]. Schematically, we may say that the structure involved is defined as follows: A first ingredient is a $\star$-product with deformation parameter $\hbar$ and the associated formal algebra of the power series. A second ingredient is another star-product with deformation parameter $\beta = \frac{1}{T}$ that defines a new product on the formal algebra. A third ingredient is a *conformal symplectic structure*, defined by β, on some manifold [3]. In physics, the new parameter β denotes the inverse of the temperature T of the system. Given this algebraic and geometric setting, a suitable internal automorphism of the algebra of observables can now be defined (in particular, by complexifying t to $t + i\hbar\beta$) by $\exp(-(t + i\hbar\beta)H)$. This automorphism then serves to define the *physical* classical and quantum KMS states, or KMS boundary condition, in terms of $\star$-products. Here H is a classical observable.

Since its introduction in [4, 5], the $\star$-*exponential* function (or distribution) has played an important role in a variety of problems, such as $\star$-representation theory of Lie groups. Also, [7] solves the problem of determining whether the mathematical equivalence of two star-products is the same notion as that of the equivalence of their quantizations. Looking at the boundary value aspect may prove useful in all such domains, but we shall not develop this further here. Nevertheless, remaining at the computational level, it is worth mentioning that, in particular, [6] contains interesting developments from which further computations, similar to those shown here, can be achieved in the $2n$-dimensional case, $\mathbb{R}^n \times \mathbb{R}^n$.

2. The Moyal $\star$-product and its infinitesimal automorphisms

Let $(V_{2n};\omega)$ be a symplectic manifold. Let us assume that it is non-compact and connected. Any $\star$-product on this manifold has a finite-dimensional Lie algebra $L_{\mathscr{A}}$ of infinitesimal automorphisms. Whenever the $\star$ -product is canonical, i.e., defined by a flat, symmetric, symplectic connection on the manifold, $L_{\mathscr{A}}$ coincides with the affine symplectic Lie algebra of the triple $(V_{2n};\omega;\Gamma)$, where Γ is the form of the connection [4]. A Hamiltonian $a \in C^{\infty}(V_{2n})$ corresponding to the vector field $X_a \in L_{\mathscr{A}}$ verifies (denoting by P the *Poisson bracket* defined by ω)

$$P(a;f\star g) = P(a;f)\star g + f\star P(a;g). \tag{6}$$

Any canonical $\star$-product also satisfies the equality

$$a\star b - b\star a = i\hbar P(a;b), \quad \forall a,b \in \mathscr{A}. \tag{7}$$

Throughout this paper, $\hbar$ denotes the deformation parameter in the $\star$-product.

Definition 1. ([4]) *A $\star$-product on the manifold $(V_{2n};\omega)$ is said $L_{\mathscr{A}}$-invariant, or invariant under the Poisson Lie algebra $\mathscr{A}$, if it verifies* (6) *and* (7).

For simplicity we consider only the case where $n = 1$. See [6] for $n > 1$. Let $(\mathbb{R}\times\mathbb{R};\omega)$ be the symplectic manifold, $\psi = (\psi_1;\psi_2)$ a generic point, and $\omega(\psi;\eta) = \psi_1\eta_2 - \psi_2\eta_1$, the symplectic form. Let f,g be two elements in $\mathbb{C}^{\infty}(\mathbb{R}\times\mathbb{R})$. The Moyal $\star$-product [4] on this manifold is

$$(f\star g)(\xi) = \sum_{k=0}^{\infty}\left(\frac{i\hbar}{2}\right)^k \frac{1}{k!}P^k(f;g)(\xi), \tag{8}$$

where $P^k(f;g) = \left(\partial_1\otimes\partial_2 - \partial_2\otimes\partial_1\right)^k(f;g)$ is the k-th power of P, considered as a bidifferential operator.

Expression (8) is the asymptotic expansion in powers of $\hbar$ of the integral

$$(f\star g)(\xi) = (\pi\hbar)^{-2}\int \exp^{\frac{2i}{\hbar}\left(\omega(\xi;\eta)+\omega(\eta;\zeta)+\omega(\zeta;\xi)\right)} f(\eta)g(\zeta)\,d\eta\,d\zeta. \tag{9}$$

This integral should always be understood in the sense of distributions.

When f and g are polynomials, there are only a finite number of terms in series (8) whose sum coincides with the one obtained from (9). The main property of the formal series (8) being that it is a non-trivial formal deformation of the usual product of functions on the manifold $\mathbb{R}\times\mathbb{R}$.

It is easy to see [4] that for the Moyal $\star$-product given by (8) or (9), the set of polynomials of the form

$$X(p;q) = a_1q^2 + a_2p^2 + 2a_3pq + a_4q + a_5p + a_6, \quad \forall a_i \in \mathbb{R}, \tag{10}$$

where $\xi = (p;q) \in \mathbb{R}\times\mathbb{R}$, is a Lie algebra under the Poisson bracket. It is the affine symplectic Lie algebra of Definition 1 for the manifold $(\mathbb{R}\times\mathbb{R};\omega;\Gamma)$, which corresponds to the Moyal $\star$-product on this manifold. This Lie algebra is isomorphic to a semidirect product of $\mathfrak{sl}(2)$ and $\mathfrak{h}_1$, where $\mathfrak{h}_1$ is a realization of the Heisenberg Lie algebra constituted by all of the polynomials of degree ≤ 1.

Let us write expression (10) in the following matrix form

$$X(q;p) = \begin{pmatrix} q & p & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_3 & \frac{1}{2}a_4 \\ a_3 & a_2 & \frac{1}{2}a_5 \\ \frac{1}{2}a_4 & \frac{1}{2}a_5 & a_6 \end{pmatrix} \begin{pmatrix} q \\ p \\ 1 \end{pmatrix},$$

and use the notations:

$$d = a_1a_2 - a_3{}^2, \quad \Delta = \text{determinant of the matrix}, \quad c = -\frac{\Delta}{d}. \tag{11}$$

Lemma 1 *Let $r,s \in \mathbb{N}$; $P_i^r(\xi) = \xi_i^r$, $i = 1,2$, and $b \in \mathbb{C}^\infty(\mathbb{R}\times\mathbb{R})$. Then*

$$((P_1^r P_2^s) \star b)(\xi) = \sum_{l=0}^{s}\sum_{m=0}^{r} \binom{s}{l}\binom{r}{m} \xi_1^m \xi_2^l \Big(\frac{\hbar}{2i}\frac{\partial}{\partial \xi_1}\Big)^{s-l}\Big(-\frac{\hbar}{2i}\frac{\partial}{\partial \xi_2}\Big)^{r-m} b(\xi).$$

Proof. $P_1^r P_2^s$ is a polynomial. An easy computation from (9) or (8) gives the desired expression.

Lemma 2 *Let $\tilde{b} \in \mathbb{C}^\infty(\mathbb{R}^2)$ be of the form $\tilde{b}(\xi_1;\xi_2) = b(X(\xi_1;\xi_2))$, where $b \in \mathbb{C}^\infty(\mathbb{R})$. Then*

$$X(p;q) \star b(X(p;q)) = \Big(-\hbar^2 d(X+c)\frac{d^2}{dX^2} - \hbar^2 d\frac{d}{dX} + X\Big)b(X). \tag{12}$$

Proof. We obtain the desired expression from Lemma 1 and definitions (11).

Definition 2 *We call Equation* (12) *the main equation.*

Two cases will be considered:

$$\begin{aligned} &(a)\ d < 0, \\ &(b)\ d > 0, \quad X(p_0;q_0) + c = 0,\ X(p;q) + c > 0,\ \forall (p;q) \neq (p_0;q_0). \end{aligned} \tag{13}$$

The point $(p_0;q_0)$ is uniquely determined by the equation $X(p;q) + c = 0$ under the hypothesis $d > 0$, because the determinant in (11), where a_6 is replaced by $a_6 + c$, is zero. The case $d = 0$ has no interest.

3. The main equation and the confluent hypergeometric function ${}_1F_1$

We shall solve Equation (2). According to Lemma 2, it can be written as

$$-\hbar^2 d\big((X+c)\frac{d^2}{dX^2}+\frac{d}{dX}-\frac{1}{\hbar^2 d}X\big)\pi(\lambda;X)=\lambda\pi(\lambda;X) \quad d\neq 0. \tag{14}$$

Let us introduce the new variables:

$$\rho=c+\lambda,\ Y=X+c,\ s^2=\frac{1}{\hbar^2 d},\ U=2sY,\ V(U)=e^{\frac{1}{2}U}\pi(\rho-c;Y-c). \tag{15}$$

Equation (14) then becomes

$$U\frac{d^2V}{dU^2}+(1-U)\frac{dV}{dU}-(\frac{1}{2}-\frac{s}{2}\rho)V=0, \quad d\neq 0. \tag{16}$$

The Kummer equation is [23]

$$U\frac{d^2V}{dU^2}+(b-U)\frac{dV}{dU}-aV=0, \tag{17}$$

where a and b are complex numbers. The following integral representation of the Kummer function ${}_1F_1[a;b;U]$ holds for $\Re(b)>\Re(a)>0$ [23]:

$${}_1F_1[a;b;U]=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{Uz}(1-z)^{b-a-1}z^{a-1}\,dz. \tag{18}$$

Setting $z=\frac{1}{1+e^{-t}}$ (or equivalently $2z-1=\tanh\frac{t}{2}$) in the above integral, we obtain

$$\int_0^1 e^{Uz}(1-z)^{b-a-1}z^{a-1}dz=2^{-b}e^{\frac{1}{2}U}\int_{-\infty}^{\infty} e^{\frac{1}{2}U\tanh\frac{t}{2}}\big(e^{-\frac{1}{2}t}\big)^{b-2a}\big(\cosh\frac{t}{2}\big)^{-b}dt. \tag{19}$$

The above transformation is similar to those found in [23].

We now consider (17) for the following particular values:

$$b>0,\ a=\frac{b}{2}-\frac{|s|}{2}\rho i, \text{ where } \rho\in\mathbb{R} \text{ and } s\in\mathbb{C}. \tag{20}$$

From expressions (19), (18), (20), and the substitution $t\to\frac{|s|}{2}t$, we get

$$\int_{-\infty}^{\infty} e^{\frac{1}{2}U\tanh(\frac{t}{|s|})}\big(\cosh\frac{t}{|s|}\big)^{-b}e^{-it\rho}\,dt= \\ \frac{2^{b-1}|s|\,\Gamma(\frac{b}{2}-\frac{|s|}{2}\rho i)\Gamma(\frac{b}{2}+\frac{|s|}{2}\rho i)}{\Gamma(b)}e^{-\frac{1}{2}U}\,{}_1F_1\big[\frac{b}{2}-\frac{|s|}{2}\rho i;b;U\big]. \tag{21}$$

All functions in this expression are holomorphic for the values (20) and $\forall U\in\mathbb{C}$. We can prove [19]

Lemma 3 *The function* $F(t) = e^{m\tanh t}\big(\cosh t\big)^{-b}$, *where* $m \in \mathbb{C}$ *and* $b > 0$, *is an element of the Schwartz space* $S(\mathbb{R})$.

This lemma shows that (21) is a Fourier transform on the space $S(\mathbb{R})$ for any fixed X.

4. Computation of $\mathscr{U}(t;X)$ for $d<0$

We return to expression (21). In this case, $s = \mid s \mid i$ and $\mid s \mid = \frac{1}{\hbar\sqrt{|d|}}$. By (16) and (15), this expression becomes

$$\int_{-\infty}^{\infty} (\cosh\frac{t}{\mid s\mid})^{-1} e^{i(|x|(X+c)\tanh(\frac{t}{|s|})-ct)} e^{it\lambda}\, dt = \pi(\lambda;X), \tag{22}$$

where

$$\pi(\lambda;X) = \frac{\pi \mid s \mid}{\cosh(\frac{|s|}{2}(\lambda+c))} e^{-|s|(X+c)i} {}_1F_1\big[\frac{1}{2} - \frac{\mid s\mid}{2}(\lambda+c)i; 1; 2\mid s\mid (X+c)i\big]. \tag{23}$$

We have therefore proved

Proposition 1
1) When $d<0$, *the function* $\pi(\lambda;X)$ *in* (23) *is a solution of the equation* $X * \pi(\lambda;X) = \lambda\pi(\lambda;X)$, $\lambda \in \mathbb{R}$.

2) The function $\mathscr{U}(t;X) = \big(\cosh\frac{t}{|s|}\big)^{-1} e^{i(|s|(X+c)\tanh(\frac{t}{|s|})-ct)}$ *is bounded, real analytic and* $\mathscr{U} \in S(\mathbb{R})\hat{\otimes}\mathscr{C}^\infty(\mathbb{R})$[22].

3) Equality (22) *is a partial Fourier transform, that is,*

$$((F^{-1}\hat{\otimes}I)\mathscr{U})(\lambda;X) = \frac{1}{2\pi}\pi(\lambda;X), \quad \big((F\hat{\otimes}I)\big(\frac{1}{2}\pi\big)\big)(t;X) = \mathscr{U}(t;X).$$

Proposition 2 *Let* X *be fixed. The following equalities are verified:*

$$\begin{aligned} \frac{1}{i}\frac{d}{dt}\mathscr{U}(t;X) &= X * \mathscr{U}(t;X), \\ \mathscr{U}(t;X) * \mathscr{U}(t';X) &= \mathscr{U}(t+t';X), \quad \forall t,t' \in \mathbb{R}. \end{aligned} \tag{24}$$

Let $t_0 \neq \frac{\pi|s|}{2}$. *The series* $\mathscr{U}(t;X) = \sum_0^\infty \big(\frac{1}{n!}(i(t-t_0))^n (X*)^n * \mathscr{U}(t_0;X)\big)$ *converges in a neighborhood of* t_0.

Proof. We get the first equality from Lemma 2 and after some easy computations. The second equality follows from (9) when we exploit the particular form of $X(p;q)$. The third equality thereby becomes obvious.

5. From the case $d < 0$ to the case $d > 0$

When $d > 0$, the function $X(p;q)+c$ is positive and has only one zero, (see (13)). We will now solve the equation (or (2) and (14))

$$-\hbar^2 d\big((X+c)\frac{d^2}{dX^2}+\frac{d}{dX}-\frac{1}{\hbar^2 d}X\big)\mathscr{U}(t;X)=\frac{1}{i}\frac{d}{dt}\mathscr{U}(t;X).$$

We perform in this equation the formal substitutions

$$(t\to it;\ \ X\to iX;\ \ c\to ic;\ \ i\to -i)\equiv(\mathbf{S}),$$

so it becomes

$$-\hbar^2(-d)\big((X+c)\frac{d^2}{dX^2}+\frac{d}{dX}-\frac{1}{\hbar^2(-d)}X\big)\mathscr{U}(\mathbf{S})=\frac{1}{i}\frac{d}{dt}\mathscr{U}(\mathbf{S}).$$

This equation is identical to the one we solved for the case of $d < 0$, (14), (24), and Proposition 1. The solution that will be the most interesting for us is

$$\mathscr{U}(\mathbf{S})=\big(\cosh\frac{t}{|s|}\big)^{-1}e^{i(|s|(X+c)\tanh(\frac{t}{|s|})-ct)},$$

that is to say,

$$\mathscr{U}(t;X)=\big(\cos\frac{t}{|s|}\big)^{-1}e^{i(|s|(X+c)\tan(\frac{t}{|s|})-ct)}. \tag{25}$$

The singularities of this function are located at the points $t=(2k+1)\frac{\pi}{2}\mid s\mid$, $k\in\mathbb{R}$. We need to find out what kind of generalized function this would be on $\mathbb{R}$. For that purpose, we may consider its analytic continuation in t (or in $(t;X)$ but after Proposition 3 below, it will not be hard to see that only the former actually matters):

$$\mathscr{U}(z;X)=\big(\cos\frac{z}{|s|}\big)^{-1}e^{i(|s|(X+c)\tan(\frac{z}{|s|})-cz)},\quad \text{where } z=t+iy. \tag{26}$$

This function has no other singularities than the points $t=(2k+1)\frac{\pi}{2}\mid s\mid$, $k\in\mathbb{R}$, just mentioned.

6. Characterization of $\delta_+\mathscr{U}(t;X)$ and $\delta_-\mathscr{U}(t;X)$

Let $\Gamma^+=\{z\mid\Im(z)>0\}$ and $\Gamma^-=\{z\mid\Im(z)<0\}$, and fix X for the time being. The function (25) can be considered as *either* of the following two boundary values [10, 11, 15, 17] of the analytic function (26):

$$\delta_+\mathscr{U}(t;X)=\lim_{y\downarrow 0}\mathscr{U}(z;X);\qquad \delta_-\mathscr{U}(t;X)=\lim_{y\uparrow 0}\mathscr{U}(z;X). \tag{27}$$

The limits are taken with respect to the topologies of the spaces involved (see below). The characterization of distributions as boundary values of holomorphic functions is given in [10, 15] and that of ultradistributions in [10, 11]. Let $(a;b)$, $(0;c)\subseteq\mathbb{R}$. Let F be an analytic function on the open set $(a;b)\times i(0;c)$.

Theorem 1. **([10, 15, 16])** *The hyperfunction $\delta_+ F$ on the interval $(a;b)$ is a distribution on this interval if and only if, for each compact $K \subset (a;b)$, there is an integer $N > 0$ and a constant $C > 0$ such that, $\forall t \in (a;b)$ and small $|\, y\,|$, we have $\sup_{t\in K} |F(t+iy)| \leq C\,|\, y\,|^{-N}$. If this is the case, $\delta_+ F(t) = \lim_{y\downarrow 0} F(t+iy)$, the limit being taken in the space of distributions $\mathscr{D}'((a,b))$.*

The theorem holds for holomorphic functions of an arbitrary number of complex variables [15, 16]. Let $(a;b)$, $(d;0) \subseteq \mathbb{R}$. Let F be an analytic function on the open $(a;b) \times i(d;0)$. The following theorems are well known.

Theorem 2. **([10, 11])** *The hyperfunction δ_F on the interval $(a;b)$ is a Beurling-type, Gevrey-class ultradistribution of order $s > 1$ on this interval if and only if, for every compact $K \subset (a;b)$, there are constants $L > 0$ and $C > 0$ such that, $\forall t \in (a;b)$ and small $|\, y\,|$, $[\sup_{t\in K} |F(t+iy)| \leq C \exp\left(\left(\frac{L}{y}\right)^{\frac{1}{s-1}}\right)$. If this is the case, $\delta\_F(t) = \lim_{y\uparrow 0} F(t+iy)$, the limit being taken in the topology of the space of Beurling-type, Gevrey-class ultradistributions $\mathscr{D}^{(s)\prime}((a;b))$ of order $s > 1$.*

This theorem holds for holomorphic functions of an arbitrary number of complex variables [10, 11, 17]. We now examine the limits in (27). By so doing we get

Proposition 3

$$\left|\cos\left(\frac{t+iy}{|\,s\,|}\right)\right|^2 = \frac{1}{2}\left(\cosh\left(\frac{2y}{|\,s\,|}\right) + \cos\left(\frac{2t}{|\,s\,|}\right)\right),\ \ \forall t, y \in \mathbb{R}.$$

$$\frac{1}{\left|\cos\left(\frac{t+iy}{|s|}\right)\right|^2} \leq \frac{2}{\sum_{p\geq 1}\left(\frac{2y}{|s|}\right)^{2p}\frac{1}{(2p)!}} < \frac{|\,s\,|^2}{|\,y\,|^2},\ \ \forall t \in \mathbb{R} \text{ and small } |\,y\,|\,.$$

$$\tan\left(\frac{t+iy}{|\,s\,|}\right) = \frac{1}{\left|\cos\left(\frac{t+iy}{|s|}\right)\right|^2}\frac{1}{2}\left(\sin\left(\frac{2t}{|\,s\,|}\right) + i\sinh\left(\frac{2y}{|\,s\,|}\right)\right),\ \ \forall t, y \in \mathbb{R}.$$

From Proposition 3 and Theorem 1, we can state

Proposition 4 *$\forall t \in \mathbb{R}$, $y > 0$, small $|\,y\,|$, and fixed X, there is a constant k such that $|\mathscr{U}(t+iy;X)| < k\left(\frac{|y|}{|s|}\right)^{-1}$. Therefore $\delta_+\mathscr{U}(t;X) = \lim_{y\downarrow 0}\mathscr{U}(t+iy;X)$, the limit is taken in the topology of $\mathscr{D}'(\mathbb{R})$.*

From Proposition 3 and Theorem 2, we also have

Proposition 5 *$\forall t \in \mathbb{R}$, $y < 0$, small $|\,y\,|$, and fixed X there is a constant $\tilde{k}$ such that $|\mathscr{U}(t+iy;X)| < e^{\frac{\tilde{k}}{|y|}}$. Therefore $\delta\_\mathscr{U}(t;X) = \lim_{y\uparrow 0}\mathscr{U}(t+iy;X)$, the limit being in the space of Beurling-type, Gevrey-class ultradistributions of order $s = 2$, $\mathscr{D}^{(2)\prime}(\mathbb{R})$, [11]. $\delta\_\mathscr{U}$ is not a distribution.*

As a consequence one can prove [11, 15, 18, 19]

Proposition 6
i) $\delta_+\mathscr{U}(t;X) = \lim_{y\downarrow 0}\mathscr{U}(t+iy;X) \in \mathscr{D}'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$ *where the limit is taken in the topology of* $\mathscr{D}'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.
ii) $\delta_-\mathscr{U}(t;X) = \lim_{y\uparrow 0}\mathscr{U}(t+iy;X) \in \mathscr{D}^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$, *where the limit is taken in the topology of* $\mathscr{D}^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.

Proposition 7 *The following equalities hold:*
i) $\frac{1}{i}\frac{d}{dz}\mathscr{U}(z;X) = X * \mathscr{U}(z;X), \forall z \neq \frac{\pi|s|}{2}, k \in \mathbb{R}$
ii) $\frac{1}{i}\frac{d}{dt}\delta_+\mathscr{U}(t;X) = X * \delta_+\mathscr{U}(t;X)$.
iii) $\frac{1}{i}\frac{d}{dt}\delta_-\mathscr{U}(t;X) = X * \delta_-\mathscr{U}(t;X)$.
iv) $\delta_\pm\mathscr{U}(t;X) * \delta_\pm\mathscr{U}(t';X) = \delta_\pm\mathscr{U}(t+t';X)$.

Proof. Statement i) can be proved from the definition (26) of $\mathscr{U}(z;X)$ and the fact that the *differential operator* $(X*)$ is acting on functions that depend only on X. For statement ii), note that the operator $\frac{1}{i}\frac{d}{dt} - I\hat{\otimes}(X*)$ acts continuously on the space $\mathscr{D}'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$. Similarly, statement iii) follows from the fact that the operator $\frac{1}{i}\frac{d}{dt} - I\hat{\otimes}(X*)$ acts continuusly on the space $\mathscr{D}^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$. For statement iv), we have

$$\begin{aligned}\delta_\pm\mathscr{U}(t;X) * \delta_\pm\mathscr{U}(t';X) &= \lim_{y;y'}\big(\mathscr{U}(t+iy;X) * \mathscr{U}(t+iy';X)\big)\\ &= \lim_{y+y'}\mathscr{U}((t+t')+i(y+y');X) = \delta_\pm\mathscr{U}(t+t';X).\end{aligned}$$

The first equality is clearly the meaning of (9) for $\delta_\pm\mathscr{U}(t;X) * \delta_\pm\mathscr{U}(t';X)$.

7. Computation of the Fourier transforms $(F\hat{\otimes}I)\delta_\pm\mathscr{U}$

The following spaces will be used in this section [8]: $Z(\mathbb{R})$ denotes the topological vector space whose elements are the Fourier transforms of all the elements in $\mathscr{D}(\mathbb{R})$, endowed with the topology making the Fourier transform an isomorphism of the topological vector spaces: $\mathscr{D}(\mathbb{R}) \simeq Z(\mathbb{R})$. $Z(\mathbb{R}) \subset S(\mathbb{R})$. And the image of this continuous inclusion is dense. $Z'(\mathbb{R})$ will denote the strong dual of space $Z(\mathbb{R})$. The Fourier transform is a topological vector space isomorphism from space $\mathscr{D}'(\mathbb{R})$ to space $Z'(\mathbb{R})$: $\mathscr{D}'(\mathbb{R}) \simeq Z'(\mathbb{R})$. $S'(\mathbb{R}) \subset Z'(\mathbb{R})$, and the image of this continuous inclusion is dense.

Let $\mathscr{D}^{(2)\prime}(\mathbb{R})$ be the space of Beurling-type, Gevrey-class ultradistributions of order $s = 2$. $Z^{(2)}(\mathbb{R})$ denotes the topological vector space whose elements are the Fourier transforms of all ultradifferentiable functions of the said class and type [10]. $\mathscr{D}^{(2)}(\mathbb{R}) \subset \mathscr{D}(\mathbb{R})$ is a continuous inclusion with a dense image. $Z^{(2)}(\mathbb{R})$ is endowed with the topology making the Fourier transform an isomorphism of the topological vector spaces: $\mathscr{D}^{(2)}(\mathbb{R}) \simeq Z^{(2)}(\mathbb{R})$. $Z^{(2)\prime}(\mathbb{R})$ denotes the strong dual topological vector space of space $Z^{(2)}(\mathbb{R})$. The Fourier transform is a topological vector space isomorphism from the space $\mathscr{D}^{(2)\prime}(\mathbb{R})$ to the space $Z^{(2)\prime}(\mathbb{R})$: $\mathscr{D}^{(2)\prime}(\mathbb{R}) \simeq Z^{(2)\prime}(\mathbb{R})$. $Z'(\mathbb{R}) \subset Z^{(2)\prime}(\mathbb{R})$, $\mathscr{D}'(\mathbb{R}) \subset \mathscr{D}^{(2)\prime}(\mathbb{R})$, the images of these continuous inclusions being dense.

From the definition of the Fourier transform [12, 13], we have

$$(F\hat{\otimes}I)\delta_{\pm}\mathscr{U} = \lim_{y\downarrow 0;y\uparrow 0} I_y, \text{ where } I_y(\lambda;X) = \int_{-\infty}^{\infty} \mathscr{U}(t+iy;X)e^{-i\lambda t}\,dt, \tag{28}$$

and the limit is in $Z'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$ for $y > 0$, or in $Z^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$ for $y < 0$. We can also write

$$I_y(\lambda;X) = \mid s \mid \sum_{K=-\infty}^{K=\infty} e^{-i2K\pi|s|(\lambda+c)} \times$$
$$\int_0^{2\pi} \left(\cos\left(t' + i\frac{y}{\mid s \mid}\right)\right)^{-1} e^{i(|s|(X+c)\tan(t'+i\frac{y}{|s|}))} e^{-it'|s|(\lambda+c)} e^{cy}\,dt', \quad K \in \mathbb{Z}.$$

Now $\sum_{K=-\infty}^{K=\infty} e^{iK(-2\pi|s|(\lambda+c))} = \frac{1}{|s|}\sum_{K=-\infty}^{K=\infty} \delta_{-(\frac{K}{|s|}+c)}(\lambda)$, which converges in $S'(\mathbb{R})$ and therefore also in $Z'(\mathbb{R})$ and $Z^{(2)\prime}(\mathbb{R})$. We therefore have

$$I_y(\lambda;X) = e^{cy} \sum_{K=-\infty}^{K=\infty} \delta_{-(\frac{K}{|s|}+c)}(\lambda) J_y(K;X), \tag{29}$$

where

$$J_y(K;X) = \int_0^{2\pi} \left(\cos\left(t' + i\frac{y}{\mid s \mid}\right)\right)^{-1} e^{i(|s|(X+c)\tan(t'+i\frac{y}{|s|}))} e^{it'K}\,dt'.$$

By setting $e^{it'} = z$, the computation of this integral gives

Lemma 4

$$J_y(2n;X) = 0 \quad n = 0, \pm 1, \pm 2, \ldots$$

To compute the integral for $K = 2n+1$, $n \in \mathbb{Z}$, we write $J_y(2n+1;X) = J_y^+(2n+1;X) + J_y^-(2n+1;X)$, where + and − stand respectively for integration on Γ^+ and Γ^-. We set $z = e^{i\pi}v$ in the above integrals and thereby get $J_y(2n+1;X) = 2J_y^+(2n+1;X)$. We now need to compute

$$J_y^+(2n+1;X) = -2i \int_{z\in\Gamma^+,|z|=1} \frac{e^{-\frac{y}{|s|}}}{1+z^2e^{-\frac{2y}{|s|}}} \exp\left(\mid s \mid (X+c)\frac{z^2e^{-\frac{2y}{|s|}}-1}{z^2e^{-\frac{2y}{|s|}}+1}\right) z^{2n+1}\,dz.$$

In this integral, we write

$$a = e^{-\frac{2y}{|s|}}, \quad u = \frac{z^2a - 1}{z^2a+1}, \quad \text{and} \quad z = e^{it}. \tag{30}$$

It then becomes

$$J_y^+(2n+1;X) = -ie^{\frac{y}{|s|}(2n+1)} \int_0^{\pi} (1+u)^n (1-u)^{-(n+1)} e^{(|s|(X+c)u)} du. \tag{31}$$

We can write $u = u_1 + iu_2$. From (30), we get $u_1^2 + u_2^2 + 2\coth\left(\frac{2y}{|s|}\right)u_1 + 1 = 0$. u is thus a point in the above circumference Υ_y. We see also that the mappings $t \in [0;\pi] \xrightarrow{\tilde{y}} u \in \Upsilon_y$ are bijections. The point $u = -1$ is interior to any circle whose circumference is Υ_y for every $y > 0$, but the point $u = +1$ is not. Conversely, the point $u = +1$ is interior to any circle whose circumference is Υ_y, $\forall\ y < 0$, but the point $u = -1$ is not. We may compute (31) for $y > 0$, and then for $y < 0$, by means of the Cauchy formula applied to the contours Υ_y, $y > 0$, and Υ_y, $y < 0$.

We get, for $y > 0$,

$$J_{y>0}^+(-2(N_0+\frac{1}{2});X) = 2\pi e^{-\frac{2y}{|s|}(N_0+\frac{1}{2})}(-1)^{N_0} e^{-|s|(X+c)} L_{N_0}^0(2\mid s\mid(X+c)),$$

where $L_{N_0}^0(Y) = \frac{1}{N_0!} e^Y \left(\frac{d}{dY}\right)^{N_0}[e^{-Y}Y^{N_0}]$ are the Laguerre polynomials [21]. We can now write expression (29) as

$$I_{y>0}(\lambda;X) = 2\pi e^{cy} \sum_{N_0 \geq 0} \delta_{\left(\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)}(\lambda)\pi_{N_0}(\mid s\mid(X+c))e^{-\frac{2y}{|s|}(N_0+\frac{1}{2})}, \tag{32}$$

where $\pi_{N_0}(\mid s\mid(X+c)) = 2(-1)^{N_0}e^{-|s|(X+c)}L_{N_0}^0(2\mid s\mid(X+c)$, and the series converges in $Z'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.

Similarly, for $y < 0$, we obtain

$$I_{y<0}(\lambda;X) = 2\pi e^{cy} \sum_{N_0 \geq 0} \delta_{\left(-\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)}(\lambda)\pi_{N_0}'(-\mid s\mid(X+c))e^{-\frac{2|y|}{|s|}(N_0+\frac{1}{2})}, \tag{33}$$

where $\pi_{N_0}'(-\mid s\mid(X+c)) = 2(-1)^{N_0}e^{|s|(X+c)}L_{N_0}^0(-2\mid s\mid(X+c))$, and the series converges in $Z^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.

From (28), (32), (33), and the continuity of the Fourier transform, we deduce the following

Proposition 8

$$\delta_+\mathscr{U}(t;X) = \lim_{y\downarrow 0}\Big(\sum_{N_0} e^{i\left(\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)t}\pi_{N_0}(\mid s\mid(X+c))e^{-\frac{2y}{s|}(N_0+\frac{1}{2})}\Big). \tag{34}$$

This limit being taken in the space $\mathscr{D}'(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.

$$\delta_-\mathscr{U}(t;X) = \lim_{y\uparrow 0}\Big(\sum_{N_0} e^{i\left(-\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)t}\pi_{N_0}'(-\mid s\mid(X+c))e^{-\frac{2|y|}{s|}(N_0+\frac{1}{2})}\Big). \tag{35}$$

This limit being taken in the space $\mathscr{D}^{(2)\prime}(\mathbb{R})\hat{\otimes}\mathscr{D}'(\mathbb{R})$.

Let us write

$$u = -e^{i\frac{2}{|s|}(t+iy)}, \text{ for } y > 0, \quad \text{and} \quad u = -e^{-i\frac{2}{|s|}(t+iy)}, \text{ for } y < 0. \tag{36}$$

Proposition 9 *The series*

$$i)\ \mathscr{U}(t+iy;X) = \sum_{N_0} e^{i\left(\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)(t+iy)}\pi_{N_0}(|\,s\,|\,(X+c)), \quad y>0,$$

$$ii)\ \mathscr{U}(t+iy;X) = \sum_{N_0} e^{i\left(-\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)(t+iy)}\pi'_{N_0}(-\,|\,s\,|\,(X+c)), \quad y<0,$$

are absolutely convergent $\forall(u;(X+c)) \in \{|\,u\,| < 1\} \times \mathbb{C}$.

Proof. Let us consider the generating function of the Laguerre polynomials [21]:

$$(1-u)^{-1}e^{(-Y\frac{u}{1-u})} = \sum_{N_0} u^{N_0}L^0_{N_0}(Y).$$

This series is *absolutely convergent* in $\{|\,u\,| < 1\} \times \mathbb{C}$. By inserting the values (36) in this series, we get the series *i*) and *ii*), respectively.

As a consequence of that proposition we have

Proposition 10 *Let* $X \in \mathbb{R}$ *be fixed. At any point* $t \neq |\,s\,|\,(2n+1)\frac{\pi}{2}$, $\delta_{\pm}\mathscr{U}(t;X)$ *are holomorphic functions, and the series* $\sum_{N_0} e^{i(\frac{2}{|s|}(N_0+\frac{1}{2})-c)t}\pi_{N_0}(|\,s\,|\,(X+c))$ *and* $\sum_{N_0} e^{i\left(-\frac{2}{|s|}(N_0+\frac{1}{2})-c\right)t}\pi'_{N_0}(-\,|\,s\,|\,(X+c))$ *are Abel summable to* $\delta_{+}\mathscr{U}(t;X)$ *and* $\delta_{-}\mathscr{U}(t;X)$*, respectively. In particular, the equalities* $1 = \sum_{N_0}\pi_{N_0}(|\,s\,|\,(X+c))$ *and* $1 = \sum_{N_0}\pi'_{N_0}(-\,|\,s\,|\,(X+c))$ *are verified in the Abel sense.*

Proposition 11 *The following equalities hold:*

$$X * \pi_{N_0}(|\,s\,|\,(X+c)) = \left(\frac{2}{|\,s\,|}(N_0+\frac{1}{2})-c\right)\pi_{N_0}(|\,s\,|\,(X+c)), \tag{37}$$

$$X * \pi'_{N_0}(-\,|\,s\,|\,(X+c)) = \left(-\frac{2}{|\,s\,|}(N_0+\frac{1}{2})-c\right)\pi'_{N_0}(-\,|\,s\,|\,(X+c)), \tag{38}$$

$$\pi_{N_0}(|\,s\,|\,(X+c)) * \pi_{M_0}(|\,s\,|\,(X+c)) = \delta_{N_0M_0}\pi_{N_0}(|\,s\,|\,(X+c)),$$

$$\pi'_{N_0}(-\,|\,s\,|\,(X+c)) * \pi'_{M_0}(-\,|\,s\,|\,(X+c)) = \delta_{N_0M_0}\pi'_{N_0}(-\,|\,s\,|\,(X+c)).$$

Proof. We get the first two equalities from the second-order differential operator $X*$ and the properties of the Laguerre polynomials. The third equality is in [5] and we obtain the fourth in a similar way.

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Induced action in QCD[‡]

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Abstract. We consider the problem of fermionic composites in the strong coupling regime of the low mass scale and not-so-low mass scale QCD. The group-theoretical approach permits easily to identify those collective variables which describe composite particles in the lowest scale part of QCD. When scale increases and in absence of new approximate symmetry, we propose to study possibilities opened by induced actions for background fields with finite composite scale which acquire dynamics due to vacuum fluctuations of quarks. We investigated an induced action for scalar mesons and found that their masses can be enhanced above double quark mass, due to condensates.

Keywords: quantum anomalies, effective lagrangians

Mathematics Subject Classifications (2000): 81T50,81T10

Dedicated to the memory of Moshé Flato

We first met Moshé in Paris. For one of us it happened during his short stay at Orsay in 1969, the other was brought to Moshé in 1974 by his father, who at that time was "le Directeur à l'UNESCO."

Moshé was a phenomenon. Nature gave him so many personal qualities at extreme that rarely can be found in one human being. His energy was legendary. He was exceptionally quick-minded and intelligent. Moshé irradiated such warmth and compassion that the moment you met him you become his friend.

People loved Moshé not only because he was sympathetic, clever and really truthful to friends, but also because he belonged to the type of personalities who are so needed for active life of the scientific community. His attitude to Science was close a religious one. He was always ready to fight for Science, he considered the scientific community as a great family and felt himself responsible for its health. People admired his way of doing it with such dedication and frankness, which, they felt, they were not able to exercise themselves.

LMP was one of his creations, together with Daniel. They were first to realize that such type of journal was needed. One can find in LMP the beginning of many research lines, and among them, one of the first presentation of ideas summarized below.

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1. Introduction

The group-theoretical approach to composite particles in QCD starts from quantum anomalies. Adler-Bardeen-Gross-Jackiw chiral anomalies [1, 2, 3] were first studied perturbatively in terms of Feynman diagrams. In the formalism of functional integrals these anomalies can be obtained by integration over fermion fields [4]. Because of anomalies, the invariance properties of the quantum theory differ from those of the same theory at the classical level. As it was shown by Fujikawa [4], an anomaly arises when the classical action is invariant under the chiral group $\mathscr{H}$ of transformations, while the functional measure is not. The anomaly can be integrated up to a chiral action, which can be divided into topological and normal-parity parts. The topological part (the Wess-Zumino-Witten term [5, 6]) presents an obstruction to renormalization. However, this part can be eliminated by choosing a special set of fermionic representations. The normal-parity part represented by polynomials of fields gives the first approximation to the low energy effective Lagrangian. Thus, the chiral action summarizes all non-invariance effects of the functional integral with respect to $\mathscr{H}$.

The conformal anomaly arises as a response of bosonic and fermionic functional integrals to conformal transformations of fields in the classical action associated with the transformation $g\colon\ g_{\mu\nu} \to g_{\mu\nu} \exp(2\sigma(x))$ [7] of an external metric $g_{\mu\nu}$. By integrating the conformal anomaly in the case of fermionic and/or scalar fields, one can obtain the conformal action (while the case of gauge fields requires a special study of gauge independence [8]), which summarizes all non-invariance effects of the functional integral with respect to the conformal group.

In the group-theoretical approach both cases can be treated in a similar manner, because what is only important is a possibility to subdivide the functional integral into two factors, $Z = Z_{inv} Z_{non}$, where Z_{inv} is invariant, while Z_{non} is non-invariant under the transformations of a group $\mathscr{G}$ with the invariant Haar measure. Then invariant factors Z_{inv} and Z_{inv}^{-1} can be built separately by transforming $Z \to Z^g$, $g \in \mathscr{G}$, and integrating over all group transformations $g(x)$ — $Z_{inv} = \int DgZ^g$, or $Z_{inv}^{-1} = \int Dg\,(Z^g)^{-1}$ [9, 10]. An effective action $W(g)$ for a field $g(x)$ is defined by

$$Z_{non} \equiv \int Dg \exp iW(g), \quad W(g) = -i \ln\left(\frac{Z}{Z^g}\right) \qquad (1)$$

up to $\mathscr{G}$-invariant terms. Note that this definition does not require $\mathscr{G}$-invariance of the classical Lagrangian. Z_{non} can be considered as a functional integral for a quantum field $g(x)$ with the classical action $W(g)$ which describes symmetry breaking, i.e., chiral symmetry breaking or breaking of the conformal symmetry. Thus, it is reasonable to use such a description of composites in the scale region where the chiral and conformal symmetries breaking dominate in the physical picture (the lowest mass scale region). Characteristic phenomenological scales are given by the quark C_q and gluon C_g condensates, $C_q = -(180\text{ MeV})^3$, $C_g = (330\text{ MeV})^4$.

The group-theoretical approach was used for a systematic derivation of effective Lagrangians in the following cases:

(a) For the chiral field, $g \to U = \exp(i\Pi)$, where the pseudoscalar field Π describes pions and kaons [10];
(b) For an extended chiral field $g \to \mathscr{U}$ of the extended chiral group $E\chi$ describing pseudoscalar mesons and diquarks [11];
(c) For the dilaton field $\sigma(x)$, where $g \to \exp(\sigma)$ is related to the conformal transformation $g_{\mu\nu} \to (\exp 2\sigma) g_{\mu\nu}$ [12].

In this way we get quantum composite fields $U, \mathscr{U}$, and σ with their vacuum functionals, i.e., with integration over these fields.

The induced action for a background field describes its dynamics arising due to interaction with vacuum fluctuations of quantum fields. The question is to what extent the induced action approach can be used as a kind of extension of the group-theoretical approach into not-so-low mass scale region of QCD, when we include impact of quark and gluon condensates into an induced action and correlate the compositeness scale with the chiral breaking scale. One of the problem to solve is how do mesons in not-so-low scale region get masses exceeding two constituent quark masses (excited mesons).

In this paper we consider the simplest model of quantized quark fields in a background of scalar field and investigate properties of dynamical scalar (classical) fields resulting after integration over quarks and application of the compositeness scheme.

2. Quantum field in classical background and classical action for composite field

2.1. Induced classical gravity

The first induced action for background field has been considered in relation with gravity. Quantum matter fields on curved space represent the most general example of such class. Vacuum fluctuations of matter fields transform a given metric into a classical dynamical gravitational field with the Einstein action in the long-wave region which got the name of 'Induced Gravity'. This fact was discovered by Zeldovich and Sakharov [13] and developments were reviewed by Adler [14]. Due to an ambiguity [15] in the definition of the Newton constant, Induced Gravity was forgotten for a decade and then reconsidered [16] with modern computational tools in the framework of bosonization approach. "Einsteinization" [17] of an external metric under the influence of collective vacuum mode of matter fields represents a phenomenon of 'partial compositeness', when quantum fields do not create a new field, but only provide an existing field with a kinetic term and interactions.

Let $L\left(\Psi, g_{\mu\nu}\right)$ be a Lagrangian of matter fields Ψ on curved background $g_{\mu\nu}$ and let $Z_\Psi\left[g_{\mu\nu}\right]$ denote the vacuum functional by

$$Z_{\Psi}\left[g_{\mu\nu}\right]=\int D\Psi \exp\left(\int dx\, L\left(\Psi,g_{\mu\nu}\right)\right). \tag{2}$$

Integration over matter Ψ in Z_{Ψ} takes into account vacuum fluctuations of the fields Ψ, which provide an external metric $g_{\mu\nu}$ by kinetic term and interactions described by an effective action $W\left(g_{\mu\nu}\right)$

$$\exp\left(iW\left(g_{\mu\nu}\right)\right)\equiv Z_{\Psi}\left[g_{\mu\nu}\right]. \tag{3}$$

It is the action $W\left(g_{\mu\nu}\right)$ that describes induced Einsteinian gravity if the evaluation of Z_{Ψ} is implemented carefully.

2.2. Scalar mesons

As an example, consider a quantum quark field ψ and in background a scalar field $S(x)$ which serves as a collective variable in a model of scalar mesons [18]. A model of flavor vector mesons was constructed earlier in a similar approach [19].

The purpose of the model is to develop a theory of isotriplet and isosinglet scalar mesons a_0 and f_0. Description of the low energy QCD in terms of the quark and gluon condensates only presupposes that the coupling constant α_S is frozen up to the chiral breaking scale. Such an approximation is not valid for higher scales. We propose to describe QCD physics for higher scales, and, in particular, scalar mesons by the composite model with the finite compositeness scale. We use an approach proposed for the model of composite electroweak bosons [20] and a definition of a composite particle in renormalization theory through radiative corrections [21].We use also similarity between low energy scalars and Higgs fields when gauge fields are also present.

Integration over fermions in the quark path integral in its simplest form is equivalent to one-loop approximation which produces polynomials of scalar fields and, consequently, may lead to scalar condensate. An advantage of the path integral approach is that it can easily take into account the quark and gluon condensates [10] and incorporate the finite compositeness scale.

Our aim is to obtain an effective Lagrangian for a dynamical composite field $\widehat{S}(x)$ through radiation corrections to the Dirac Lagrangian with an external field $S(x)$. A composite field $\widehat{S}(x)$ is defined by vanishing of its renormalization constant $z_S\left(\frac{\Lambda}{\mu}\right)\to 0$ when μ reaches the compositeness scale Λ [21]. We consider composites in the approximation of fermion loops resulting from integration over fermions in the path integrals. If $S(x)$ were elementary, its renormalization constant would be $z_S=1+\Delta$, where Δ contains logarithms.For a composite $\widehat{S}(x)$ suitable normalization leads to $z_S=\Delta$.

The Lagrangian for fermions interacting with a scalar background S and the gluon field $G_\nu = (g/2i)\, G_\nu^k \lambda_k$, and the corresponding path integral are given by

$$L = \overline{\psi}\,(\not{D} + fS)\,\psi,$$
$$\mathbf{Z}[S] = \exp iW\,(S) = \int D\overline{\psi} D\psi \exp\{i \int dx\, L\} \equiv Z_\psi\,[D+S], \tag{4}$$

where D is the Dirac operator for massless fermions with covariant derivative D_μ containing the gluon field and f is the Yukawa coupling constant. The fermion path integral $\mathbf{Z}[S]$ defines an induced action $W\,(S)$ for S in presence of the gluon field. In general, the scalar background $S\,(x)$ should be introduced into the Lagrangian together with a weight function $\Omega\,(S)$ independent of fermions, $L \to L + \Omega$, which arises from other interactions of scalar S. It is understood that all expressions for $\mathbf{Z}$ are averaged over gluons G_μ.

The quark spectrum in the gluon vacuum displays non-trivial features [22] which have escaped until now from detailed analysis and are known by their manifestation in condensation process. The quark condensate

$$C_q = \langle \overline{\psi}\,(x)\,\psi\,(x)\rangle_{\overline{\psi},\psi,G} = -i \left\langle \frac{\delta \ln \mathbf{Z}}{\delta fS}\,|_{S=0} \right\rangle_G \tag{5}$$

is the vacuum expectation value obtained by averaging over both quark and gluon fields. Its numerical value is known from experiments. The origin of the quark condensate may be due to a gap in the quark spectrum density at the origin [23] or to an asymmetry of the spectral density [10, 24]. We shall deal with the quark spectrum in the Θ-function in a phenomenological way: We do not take into account the spectral gap or the asymmetry, but just integrate this expression for a given C_q and add the condensate term $C_q tr_f S$ to the induced Lagrangian $L\,(S,\Lambda)$ in the Minkowski space

$$\begin{aligned} L\,(S,\Lambda) - L\,(S,\mu) =& a\sigma f^2 tr_f\,(D_\mu S)^2 + 6a\sigma\xi_2 f^2\Lambda^2 tr_f S^2 \\ & - a\sigma f^4 tr_f S^4 - 3\sigma\xi_3 f C_q tr_f S - \sigma\frac{N_f}{24\pi^2} tr_C G_{\mu\nu}^2, \end{aligned} \tag{6}$$

where $G_{\mu\nu}$ is the gluonic field strength, ξ_n describes running of dimensionful coefficients

$$\xi_n = \frac{1 - e^{-n\sigma}}{n\sigma}, \quad \sigma = \ln\frac{\Lambda}{\mu}, \tag{7}$$

tr_f and tr_C refer to flavor and color matrices. $a = N_C/8\pi^2$, N_C is the number of colors. In (6) the kinetic energy is positive for $\sigma > 0$ and disappears at $\sigma = 0$. It follows from (6) that the induced potentials $U\,(S,\Lambda)$ and $U\,(S,\mu)$ include S^2- and S^4-terms; the S^2-term enters with a wrong sign, so that the potential may have a minimum at S_0 corresponding to formation of scalar condensate. We choose S_0 as the lower scale of the region under consideration with the compositeness scale

Λ_S as the upper scale. The normalization scale μ is arbitrary within the region $S_0 \leq \mu \leq \Lambda_S$.

Note that with the specification of the path integral by an invariant momentum cut-off, the Lagrangian (6) does not contain terms with negative powers of the cut-off Λ. The term proportional to $\Lambda^2 - \mu^2$ is absent, because it is always understood that the free fermion part is eliminated.

With the standard kinetic term in the form $\frac{1}{4} tr_f \left(D_\mu S_r\right)^2$ for the renormalized field S_r we get the following renormalization constant and couplings

$$z_S = 4a\sigma f^2, \quad m^2 = \frac{3}{2}\xi_2\Lambda^2, \quad \lambda = \frac{1}{4a\sigma}, \quad \eta = 3\sqrt{\frac{\sigma}{a}}\xi_3 \tag{8}$$

for the potential

$$U^r_{eff} = -m^2 tr S_r^2 + \frac{\lambda}{4} tr S_r^4 - \eta C_q tr S_r. \tag{9}$$

Thus, at the compositeness scale $\sigma \to 0$, the coupling λ is diverging, m^2 remains finite, while η becomes inessential. By neglecting η we would get for a flavor singlet S_r the minimum of U^r_{eff} for $\sigma \to 0$ at $S_r^2 = (3N_C/4\pi^2)\Lambda^2\sigma \ll \Lambda^2$, i.e., well below the upper scale.

In order to study composite scalars far from the compositeness scale, one should use the induced action $W(S,\mu)$. The easiest way to find $W(S,\mu)$ is to calculate separately the potential and the kinetic terms. The potential (even powers of S and logarithms) can be calculated exactly; the quark condensate term will be neglected. For the kinetic term one can use an expansion in the derivatives. We get for the induced Lagrangian

$$\begin{aligned} L(S,\mu) = &\frac{1}{2} a f^2 tr_f \ln(\frac{\mu^2}{f^2 S^2}) (D_\nu S)^2 + 3 a f^2 tr_f \left(\mu^2 - f^2 S^2\right) S^2 - \frac{1}{4} a f^4 tr_f S^4 \\ &- \frac{1}{2} a f^4 tr_f S^4 \ln(\frac{\mu^2}{f^2 S^2}) - \frac{N_f}{48\pi^2} tr_C(G^2_{\mu\nu}) tr_f \ln\left(\frac{\mu^2}{f^2 S^2}\right). \end{aligned} \tag{10}$$

under the condition $\Lambda_S^2 \geq \mu^2 \geq f^2 S^2$. For the compositeness scale $\mu = \Lambda_S$, the induced potential $U\left(S, \Lambda_S\right)$ will have a minimum at a point $S = S_0$ defining the lowest scale in the renormalization constant $z_S = 2af^2 \ln\left(\Lambda_S^2/f^2 S_0^2\right)$ for $L(S,\Lambda_S)$.

In the model of induced action with the finite compositeness scale Λ_s, we assune that $L(S,\mu) = L(\mu,\Lambda_s)$ and consider

$$L(S,\Lambda_s) = \frac{1}{2}(L(S,\mu) + L(\mu,\Lambda_s)) \tag{11}$$

as an effective action for the model.

3. Nonlinear representation of the composite scalar field

The most interesting case is when the composite scalar field develops condensate and interacts with the quark condensate. Therefore, we consider a nonlinear representation for $S(x)$ and denote it, in this case, by $\Phi = \Phi_0 \exp\phi$, where Φ_0 is a large slowly varying field and $\phi = \phi_0 + \phi_k \tau_k$ are small fluctuations; τ_k are the Pauli matrices for isospin.

The induced Lagrangian for the renormalized composite field $\Phi_r = z_\phi^{1/2}\Phi$, $z_\phi = z_S$, and the compositeness scale Λ_s as the higher scale μ is given by

$$L(\Phi_r,\Lambda_S) = \frac{1}{4} tr_f (D_\nu \Phi_r)^2 - U\left(\Phi_r,\Lambda_S\right), \quad a = 0,k, \tag{12}$$

where U is the induced potential with positive coefficients $m_0^2, \lambda, \zeta, \varepsilon^4$ defined at $\mu = \Lambda$:

$$U\left(\Phi_r,\Lambda_S\right) = -m_0^2 tr_f \Phi_r^2 + \frac{\lambda_0}{4} tr_f \Phi_r^4 + \frac{\zeta}{4} tr_f \Phi_r^4 \ln\left(\frac{\Lambda_S^2}{f^2\Phi_r^2}\right) + \varepsilon^4 \ln\frac{\Lambda_S^2}{f^2\Phi_r^2}, \tag{13}$$

with the minimum at Φ_0 defined by the equations

$$\left(2\lambda_0 - \zeta\right)\Phi_0^4 + 2\zeta\Phi_0^4 \ln\frac{\Lambda_S^2}{\Phi_0^2} - 4m_0^2\Phi_0^2 - 4\varepsilon^4 = 0, \tag{14a}$$

$$(2\lambda_0 - 3\zeta + 2\zeta\ln\frac{\Lambda_S^2}{\Phi_0^2})\Phi_0^4 + \varepsilon^4 \succ 0, \tag{14b}$$

and the mass M of the composite field by $M^2 = 8\left(\lambda\Phi_0^2 - m_0^2\right)$.

In this relation the mass M does not depend explicitly on the gluon condensate coefficient ε^4 and on the coefficient ζ. An expression for the mass looks as if these two coefficients are zero. The relation between the mass of composite field and the quark mass $z_\phi M^2/\Phi_0^2$ is a function of the parameter $\chi^2 = \Lambda_S^2/\Phi_0^2$ that behaves like $(\ln\chi)^{-1}$. Thus, it is possible to have χ larger than 2. It means that two quarks can form a composite scalar field with a mass higher than the double quark mass due to contribution from gluons through the condensate. It can be seen explicitly when we can disregard terms with the coefficient ζ, i.e., when approximately $\Lambda_S^4 = e\Phi_0^4$. Then $\lambda\Phi_0^2 - m_0^2 = \left(m_0^4 + 2\lambda\varepsilon^4\right)^{1/2}$.

4. Conclusions

In the induced action approach, classical external fields in the quantum action get kinetic terms and interactions after integration over quantum fields. Such dynamical fields can be considered as composite fields formed by quantized fields integrated out. Induced actions obtained in this way may contain usual ultraviolet and infrared singularities.

We have considered a model of the composite scalar field induced by integration over quarks on background of classical scalar field with the same quantum numbers. Induced action admits formation of scalar condensate that plays a role at the lowest scale, thus preventing appearance of infrared singularities. In order to model the behaviour of the QCD coupling constant in the region with scalar mesons, we propose to introduce a finite compositeness scale as a phenomenological parameter. Contribution from the gluon condensate enhances the mass of the composite scalar field above two quark masses.

In realistic cases one should consider weight functions for external background scalars arising from other interactions as well as composite axial vector mesons together with contribution from the pure QCD sector.

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Deformation quantization of Fréchet-Poisson algebras: Convergence of the Moyal product

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Dedicated to the memory of Professor Moshé Flato

1. Introduction

Beyond formal deformation quantization, i.e., deformation quantization of Poisson algebras with a formal deformation parameter [1], one can ask whether the formal parameter converges. In this direction, Rieffel [7] presented a notion of strict deformation quantization: Deformation quantization with a convergent product in the C^*-algebras sense. This suggests us opportunities of finding various kind of notions of deformation quantizations with a convergent product for suitable categories of algebras.

The purpose of this paper is to give a notion of deformation quantization corresponding to Rieffel's work [7] in the Fréchet categories, and to show that several strange phenomena occur when we treat exponential functions of quadratic forms.

Let $\mathscr{F}$ be a commutative, associative Fréchet algebra over $\mathbb{C}$, i.e., $\mathscr{F}$ has a metrizable complete topology defined by a family of semi-norms, and a product operation denoted by dot $\cdot$ which is continuous.

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$\mathscr{F}$ is called a *Fréchet-Poisson algebra* if $\mathscr{F}$ has a continuous bilinear operation $\{\ ,\ \} : \mathscr{F} \times \mathscr{F} \to \mathscr{F}$ (called a *Poisson bracket* on $\mathscr{F}$) such that for any $f, g, h \in \mathscr{F}$:

(P1) $\{f,g\} = -\{g,f\}$ (skew symmetry);

(P2) $\Sigma_{\text{cyclic sum}}\{f,\{g,h\}\} = 0$ (Jacobi identity);

(P3) $\{f,gh\} = \{f,g\}h + g\{f,h\}$ (bi-derivation).

Generally, if $\mathscr{F}$ is a Fréchet space and has an associative product $*$ on $\mathscr{F}$ such that the operation $* : \mathscr{F} \times \mathscr{F} \to \mathscr{F}$ is continuous, we call $(\mathscr{F}, *)$ a *Fréchet algebra.*

We now give a notion of *deformation quantization of Fréchet-Poisson algebra* as a family of associative product $*_\hbar$ on $\mathscr{F}$ parameterized by $\hbar \in \mathbb{R}$.

Definition 1.1 *Let $\hbar \in \mathbb{R}$. Let $\mathscr{F}$ be a Fréchet-Poisson algebra. $(\mathscr{F}, *_\hbar)$ is called a deformation quantization of a Fréchet-Poisson algebra $\mathscr{F}$ if the following conditions hold:*

(FD1) For any $\hbar$, there exists an associative product $*_\hbar$ on $\mathscr{F}$ so that $(\mathscr{F}, *_\hbar)$ is a Fréchet algebra.

(FD2) $f *_\hbar g$ is continuous in $\hbar \geq 0$ for every $f, g \in \mathscr{F}$ independent of $\hbar$, and $f *_0 g = f \cdot g$.

(FD3) $\frac{1}{\hbar i}(f *_\hbar g - f \cdot g) \to \frac{1}{2}\{f,g\}$ as $\hbar \to 0$ for every $f, g \in \mathscr{F}$ independent of $\hbar$.

We now give an example of the above notion, which is based on the Moyal product for functions as follows: Let $\mathbb{C}^2$ be the complex 2-plane, and x, y the coordinates on $\mathbb{C}^2$.

Denote by $\mathscr{P}(\mathbb{C}^2)$ the set of all polynomials of x and y, and denote by $\mathscr{E}(\mathbb{C}^2)$ the set of all entire functions on $\mathbb{C}^2$. $\mathscr{E}(\mathbb{C}^2)$ is a complete topological vector space under the compact open topology.

For entire functions $f = f(x,y)$ and $g = g(x,y)$ on $\mathbb{C}^2$, we set

$$\{f,g\} = \partial_y f \cdot \partial_x g - \partial_x f \cdot \partial_y g. \tag{1.1}$$

This gives a Poisson bracket on $\mathbb{C}^2$, which is canonical in the sense that the coordinates x, y turn to be Darboux coordinates.

The Moyal product formula is given as follows [5]:

$$f *_\hbar g = \sum_k \left(\frac{\hbar i}{2}\right)^k \frac{1}{k!} f(\overleftarrow{\partial}_y \cdot \overrightarrow{\partial}_x - \overleftarrow{\partial}_x \cdot \overrightarrow{\partial}_y)^k g. \tag{1.2}$$

In general, $f *_\hbar g$ is not always defined for any $f, g \in \mathscr{E}(\mathbb{C}^2)$, but the following properties hold:

- $*_\hbar$ defines an associative product on the space $\mathscr{P}(\mathbb{C}^2)$.
- For every fixed $p \in \mathscr{P}(\mathbb{C}^2)$, the left- and the right-multiplication $p*_\hbar$, $*_\hbar p$ extend to continuous linear mappings

$$p*_\hbar,\ *_\hbar p\colon \mathscr{E}(\mathbb{C}^2) \to \mathscr{E}(\mathbb{C}^2).$$

Using polynomial approximation, we see easily that the associativity

$$f *_\hbar (g *_\hbar h) = (f *_\hbar g) *_\hbar h$$

holds, if two of f, g, h are polynomials. We call such a system $(\mathscr{E}(\mathbb{C}^2), \mathscr{P}(\mathbb{C}^2); *_\hbar)$ a $\mathscr{P}(\mathbb{C}^2)$-*bimodule*.

We consider the following subspace of $\mathscr{E}(\mathbb{C}^2)$: For every positive $p > 0$, set

$$\mathscr{E}_p(\mathbb{C}^2) = \{f \in \mathscr{E}(\mathbb{C}^2) \mid \|f\|_{p,s} = \sup |f|\, e^{(-s|\xi|^p)} < \infty, \forall s > 0\}, \tag{1.3}$$

where $|\xi| = (|x|^2 + |y|^2)^{1/2}$. The family $\{\|\ \|_{p,s}\}_{s>0}$ induces a topology on $\mathscr{E}_p(\mathbb{C}^2)$ and $(\mathscr{E}_p(\mathbb{C}^2), \cdot)$ is an associative commutative Fréchet algebra, where the dot $\cdot$ is the ordinary multiplication for functions in $\mathscr{E}_p(\mathbb{C}^2)$. It is easily seen that for $0 < p < p'$, we have a continuous embedding

$$\mathscr{E}_p(\mathbb{C}^2) \subset \mathscr{E}_{p'}(\mathbb{C}^2) \tag{1.4}$$

as commutative Fréchet algebras [3].

It is obvious that every polynomial is contained in $\mathscr{E}_p(\mathbb{C}^2)$ and $\mathscr{P}(\mathbb{C}^2)$ is dense in $\mathscr{E}_p(\mathbb{C}^2)$ for any $p > 0$. The Poisson bracket (1.1) is also well-defined on $\mathscr{E}_p(\mathbb{C}^2)$, and $(\mathscr{P}(\mathbb{C}^2), \{,\}, \cdot)$ is a dense Poisson subalgebra of $(\mathscr{E}_p(\mathbb{C}^2), \{,\}, \cdot)$. We remark that every exponential function $e^{\alpha x + \beta y}$ is contained in $\mathscr{E}_p(\mathbb{C}^2)$ for any $p > 1$, but not in $\mathscr{E}_1(\mathbb{C}^2)$, and functions such as $e^{ax^2 + by^2 + 2cxy}$ are contained in $\mathscr{E}_p(\mathbb{C}^2)$ for any $p > 2$, but not in $\mathscr{E}_2(\mathbb{C}^2)$.

Our main result in this paper is as follows:

Theorem 1.2 *The Moyal product formula* (1.2) *gives the following:*

(i) *For* $0 < p \leq 2$, $(\mathscr{E}_p(\mathbb{C}^2), *_\hbar)$ *is a deformation quantization of* $(\mathscr{E}_p(\mathbb{C}^2), \cdot, \{,\})$.

(ii) *For* $p > 2$ *and a fixed* $\hbar \in \mathbb{R}$, *the Moyal product formula gives a continuous bilinear mapping of*

$$\begin{aligned} &\mathscr{E}_p(\mathbb{C}^2) \times \mathscr{E}_{p'}(\mathbb{C}^2) \to \mathscr{E}_p(\mathbb{C}^2), \\ &\mathscr{E}_{p'}(\mathbb{C}^2) \times \mathscr{E}_p(\mathbb{C}^2) \to \mathscr{E}_p(\mathbb{C}^2), \end{aligned} \tag{1.5}$$

for every p' *such that* $\frac{1}{p} + \frac{1}{p'} \geq 1$.

We remark here about the statement (ii). Since $p > 2$, p' must be $p' \leq 2$, hence the statement (i) gives that $(\mathscr{E}_{p'}(\mathbb{C}^2); *_\hbar)$ is a Fréchet algebra. So the statement (ii) means that every $\mathscr{E}_p(\mathbb{C}^2)$, $p > 2$, is a topological $\mathscr{E}_{p'}(\mathbb{C}^2)$-bimodule.

We remark also that if $\hbar > 0$, then $e^{\pm\frac{2i}{\hbar}xy} \in \mathscr{E}_p(\mathbb{C}^2)$ for every $p > 2$, but $e^{\frac{2i}{\hbar}xy} *_\hbar e^{-\frac{2i}{\hbar}xy}$ diverges. This means in a sense the above result is the best possible.

We define a complex conjugate $\bar{a}$ of $a \in \mathscr{E}(\mathbb{C}^2)$ as an anti-homomorphism of the bimodule by setting $\bar{x} = x, \bar{y} = y, \bar{i} = -i$ on generators. It is obvious that $\mathscr{E}_p(\mathbb{C}^2)$ is closed under the complex conjugation, and in general this looks as a very nice property.

However, this property causes some trouble for $p > 2$ as follows: We fix $\hbar > 0$. Then, the coordinate function y has a right-inverse $\frac{1}{y}(1 - e^{\frac{2i}{\hbar}xy})$, and a left-inverse $\frac{1}{y}(1 - e^{-\frac{2i}{\hbar}xy})$ as the *complex conjugate* of the right-inverse. If $(\mathscr{E}_p(\mathbb{C}^2); *_\hbar)$ were an associative algebra, then these would be the same genuine inverse, and we must set the nontrivial element $\frac{1}{y}\sin\frac{2}{\hbar}xy = 0$. Hence, it seems inconvenient to treat the right and the left inverse of y in the framework of C^* algebras. We discuss such phenomena more closely in sections 5 and 7.

2. Completion of the free tensor algebra

Let $\mathscr{T}$ be the free tensor algebra generated by X_1, X_2. We introduce a topology into $\mathscr{T}$ so that $\mathscr{T}$ becomes a topological algebra. By definition, an element of $\mathscr{T}$ is written in the form

$$T = \sum c_\alpha X_\alpha \quad \text{(finite sum)}$$

where $X_\alpha = X_{\alpha_1} \otimes X_{\alpha_2} \otimes \cdots \otimes X_{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. We call X_α a *word* and we denote by $|\alpha|$ the *length* n of the word X_α. We define a system of semi-norms $\| \ \|_{\tau,s}$ $(\tau \geq 0, s > 0)$ by

$$\|T\|_{\tau,s} = \sum |c_\alpha| |\alpha|^{\tau|\alpha|} s^{\tau|\alpha|}. \tag{2.1}$$

For a fixed $\tau > 0$, we consider a topology introduced by a system of semi-norms $\{\| \cdot \|_{\tau,s}\}_{s>0}$. We set $\mathscr{T}_\tau$ the completion of $\mathscr{T}$ with respect to this topology. Then one easily sees that

$$\mathscr{T}_\tau = \{T = \sum_\alpha c_\alpha X_\alpha \text{ (infinite sum)}; \ \|T\|_{\tau,s} < \infty \text{ for any } s > 0\}. \tag{2.2}$$

Then we have

Lemma 2.1 *For $\tau > 0$, $\mathscr{T}_\tau$ becomes a Fréchet algebra satisfying*

$$\|T \otimes T'\|_{\tau,s} \leq \|T\|_{\tau,2s} \|T'\|_{\tau,2s}.$$

Proof. It follows easily from the inequality

$$m^{\tau m} n^{\tau n} \leq (m+n)^{\tau(m+n)} \leq 2^{\tau(m+n)} m^{\tau m} n^{\tau n} \tag{2.3}$$

for non-negative integers m, n. ■

Furthermore, we have

Lemma 2.2 *Let* $T = \sum c_\alpha X_\alpha$ *be an element of* $\mathscr{T}$ *and let* N *be the maximum length of words* X_α *of* T. *Then* $e^T = \sum \frac{1}{k!} T^{\otimes k}$ *is an element of* $\mathscr{T}_\tau$ *for* $0 < \tau < \frac{1}{N}$.

Proof. It suffices to see the convergence of $\sum_{k=0}^{\infty} \frac{1}{k!} (kN)^{k\tau N} s^{k\tau N}$ under the condition $\tau N < 1$. Since

$$\begin{aligned} R_k = \frac{1}{(k+1)!} & \quad ((k+1)N)^{(k+1)\tau N} s^{(k+1)\tau N} / \tfrac{1}{k!} (kN)^{k\tau N} s^{k\tau N} \\ & = s^{\tau N} \tfrac{((k+1)N)^{\tau N}}{k+1} (1 + \tfrac{1}{k})^{k\tau N} \end{aligned}$$

one sees easily that $\lim_{k\to\infty} R_k = 0$ if $\tau N < 1$. ■

3. Subspaces of symmetric elements

Now, we consider a subspace $\mathscr{S}_\tau \subset \mathscr{T}_\tau$ equipped with the symmetric product.

First we introduce a product $a \circ b = \frac{1}{2}(a \otimes b + b \otimes a)$ in $\mathscr{T}$. We set [5]

$$(a\circ)^p \cdot c = a \circ (a \circ (\cdots (a \circ c) \cdots))$$

and

$$(a\circ)^p \cdot (b\circ)^q \cdot c = a \circ (a \circ (\cdots (a \circ (b\circ)^q \cdot c) \cdots)).$$

Using this notation, we define a linear subspace $\mathscr{S}$ as

$$\mathscr{S} = \{\sum c_{\alpha,\beta} (X_1\circ)^\alpha \cdot (X_2\circ)^\beta \cdot 1\}.$$

We also introduce a commutative product $\odot$ on $\mathscr{S}$ by

$$\begin{aligned} ((X_1\circ)^k & \cdot (X_2\circ)^l \cdot 1) \odot ((X_1\circ)^m \cdot (X_2\circ)^n \cdot 1) \\ = & \quad (X_1\circ)^{k+m} \cdot (X_2\circ)^{l+n} \cdot 1. \end{aligned}$$

For simplicity, we write

$$X_1^\alpha \odot X_2^\beta = (X_1\circ)^\alpha \cdot (X_2\circ)^\beta \cdot 1. \tag{3.1}$$

Then the set $\mathscr{S}$ with the multiplication $\odot$ becomes a commutative associative algebra.

Let $\mathscr{S}_\tau$ be the closure of $\mathscr{S}$ in $\mathscr{T}_\tau$. It is easy to see the following:

Proposition 3.1 *For every* $\tau > 0$, *the bilinear product* $\odot$ *is continuous and* $\mathscr{S}_\tau$ *is a commutative Fréchet algebra.*

On the space $\mathscr{S}$ we consider a non-commutative product $*_\hbar$ by using the Moyal product formula with the parameter $\hbar$.

To obtain the norm estimate of the $*_\hbar$-product, we remark first the following

Lemma 3.2 *For every $\tau > 0$ and every $s >> 0$, we have an inequality*

$$\|\partial^k_{X_1}\partial^l_{X_2} X_1^m \odot X_2^n\|_{\tau,s} \le \|X_1^m \odot X_2^n\|_{\tau,s} m^{k(1-\tau)} n^{l(1-\tau)}$$

Proof. Obviously we have that $\partial^k_{X_1}\partial^l_{X_2} X_1^m \odot X_2^n = \frac{m!}{(m-k)!}\frac{n!}{(n-k)!} X_1^{m-k} \odot X_2^{n-l}$. By the definition of semi-norms, we see that

$$\begin{aligned}\|\partial^k_{X_1}\partial^l_{X_2} X_1^m \odot X_2^n\|_{\tau,s} &= \frac{m!}{(m-k)!}\frac{n!}{(n-l)!} s^{\tau(m+n-k-l)}(m+n-k-l)^{\tau(m+n-k-l)} \\ &< m^{\tau k} m^{k(1-\tau)} n^{\tau l} n^{l(1-\tau)} s^{\tau(m+n-k-l)}(m+n)^{\tau(m+n-k-l)} \\ &< m^{k(1-\tau)} n^{l(1-\tau)} s^{\tau(m+n-k-l)}(m+n)^{\tau(m+n)}.\end{aligned}$$

The desired estimate follows easily. ■

Using this estimate we have the following:

Theorem 3.3 *The product $*_\hbar$ extends to $\mathscr{S}_\tau$ for $\frac{1}{2} \le \tau$ to define $(\mathscr{S}_\tau, *_\hbar)$ a non-commutative Fréchet algebra.*

In this algebra we have also the inequality

$$\|f *_\hbar g\|_{\tau,s} \le \|f \odot g\|_{\tau, se^{|\hbar|/\tau}}. \tag{3.2}$$

Proof. For $f = \sum a_{k,l} X_1^k \odot X_2^l$, $g = \sum b_{m,n} X_1^m \odot X_2^n$, a rough estimate gives

$$\begin{aligned}\|f *_\hbar g\|_{\tau,s} \le \sum_{k,l,m,n} &|a_{k,l}|\ |b_{m,n}| s^{\tau(k+l+m+n)}(k+l)^{\tau(k+l)}(m+n)^{\tau(m+n)} \\ &\times \quad \textstyle\sum_p \frac{1}{p!}|\hbar|^p (k+l)^{p(1-\tau)}(m+n)^{p(1-\tau)}.\end{aligned}$$

Remarking

$$(k+l)^{p(1-\tau)}(m+n)^{p(1-\tau)} < (k+l+m+n)^{2p(1-\tau)}$$

and

$$\sum_p \frac{1}{p!}|\hbar|^p (k+l+m+n)^{2p(1-\tau)} = e^{|\hbar|(k+l+m+n)^{2(1-\tau)}},$$

we see that $\frac{1}{2} \le \tau$ gives $2(1-\tau) \le 1$ and hence

$$\|f *_\hbar g\|_{\tau,s} \le \|f \odot g\|_{\tau, se^{|\hbar|/\tau}}.$$

■

In the above estimate, we have no need to sum up p for all non-negative integers, but p is restricted in fact in the domain $0 \leq p \leq \min(k+l, m+n)$. Thus, if $0 < \tau < \tau'$ then we can set

$$(k+l)^{\tau(k+l)}(k+l)^{p(1-\tau)} \leq (k+l)^{\tau'(k+l)}(k+l)^{p(1-\tau')}.$$

Now suppose $0 < \tau < \frac{1}{2}$. Then we have

Lemma 3.4 *For τ' such that $0 < \tau < \tau'$ and $\tau + \tau' \geq 1$, it holds*

$$\|f *_\hbar g\|_{\tau,s} \leq \|f\|_{\tau, se^{|\hbar|/\tau}} \|g\|_{\tau', se^{|\hbar|/\tau'}}.$$

Thus, we have

Theorem 3.5 *If $0 < \tau < \frac{1}{2}$, the product $*_\hbar$ gives continuous bilinear products*

$$*_\hbar : \mathscr{S}_\tau \times \mathscr{S}_{\tau'} \to \mathscr{S}_\tau, \quad *_\hbar : \mathscr{S}_{\tau'} \times \mathscr{S}_\tau \to \mathscr{S}_\tau$$

for every τ' with $\tau < \tau'$ and $\tau + \tau' \geq 1$.

4. Two-sided ideal of the relations

Suppose $\hbar \in \mathbb{C}$. Let $\mathscr{R}$ be a two-sided ideal generated by the fundamental relation of the Weyl algebra:

$$X_1 \otimes X_2 - X_2 \otimes X_1 + \hbar i$$

Definition 4.1 *We call the quotient algebra $\mathscr{T}/\mathscr{R} = \mathscr{W}$ the Weyl algebra.*

An element of $\mathscr{S}$ is written as $T = \sum c_{\alpha,\beta}(X_1)^\alpha \odot (X_2)^\beta$. From the identity (3.1), we have the natural embedding

$$\iota : \mathscr{S} \to \mathscr{T}.$$

We consider now the replacement $\otimes$ sign in every word of an element of $\mathscr{T}$ by $*_\hbar$ sign in $\mathscr{S}$. For instance

$$\begin{aligned} X_2 \otimes X_1 \otimes X_2 \to X_2 *_\hbar X_1 *_\hbar X_2 &= X_2 \odot (X_1 *_\hbar X_2) + \frac{\hbar i}{2} X_2 \\ &= X_1 \odot X_2^2 - \frac{\hbar i}{2} X_2 + \frac{\hbar i}{2} X_2. \end{aligned}$$

Obviously X_1, X_2 are contained in $\mathscr{S}$, and this replacement gives a linear mapping φ from $\mathscr{T}$ onto $\mathscr{S}$. Since the product $*_\hbar$ is defined by using the Moyal product formula, it is not hard to see that φ is an algebra homomorphism of $(\mathscr{T}, \otimes)$ onto $(\mathscr{S}, *_\hbar)$. Remarking that $\varphi \circ \iota(X_1^m \odot X_2^n) = X_1^m \odot X_2^n$, we see that $\varphi \circ \iota =$ identity.

Recall now that Weyl algebra can be naturally identified with the space of all polynomials with the Moyal product formula [5]. Hence, we see that the kernel of φ must be the ideal $\mathscr{R}$ of the relations of Weyl algebra.

Thus, we have a linear splitting

$$\mathscr{T} = \mathscr{S} \oplus \mathscr{R}.$$

Now, we consider the closures of these spaces $\mathscr{T}_\tau$, $\mathscr{S}_\tau$ and $\mathscr{R}_\tau$. By Proposition 3.1 and inequality (3.2), we have the following:

Theorem 4.2 *Suppose that* $\tau \geq \frac{1}{2}$. *Then,* φ *and* ι *extend to continuous mappings* $\varphi : \mathscr{T}_\tau \to \mathscr{S}_\tau$ *and* $\iota : \mathscr{S}_\tau \to \mathscr{T}_\tau$. *Hence if* $\tau \geq \frac{1}{2}$, *then* $\mathscr{T}_\tau = \mathscr{S}_\tau \oplus \mathscr{R}_\tau$.

5. Collapsing of algebras

In this section, we show the following:

Theorem 5.1 *If* $0 < \tau < \frac{1}{2}$, *then* $\mathscr{T}_\tau = \mathscr{R}_\tau$.

The above theorem is obtained by showing $\mathscr{R}_\tau \ni 1$ for $0 < \tau < \frac{1}{2}$.

Recall that for any $\tau > 0$, $\mathscr{R}_\tau$ is a closed two-sided ideal of $\mathscr{T}_\tau$ and the quotient algebra $\mathscr{T}_\tau / \mathscr{R}_\tau$ is a topological associative algebra.

Suppose $\mathscr{T}_\tau / \mathscr{R}_\tau$ is not a trivial algebra. $\mathscr{T}_\tau / \mathscr{R}_\tau$ is then an associative Fréchet algebra. Since the Weyl algebra $\mathscr{W}$ is simple and is identified with $\mathscr{T}/\mathscr{R}$, $\mathscr{W}$ is densely embedded in $\mathscr{T}_\tau / \mathscr{R}_\tau$.

Recall that $\mathscr{W}$ is also identified with the space of all polynomials with the product given by Moyal product formula. Thus, it is identified with $\mathscr{S}$ with the Moyal product $*_\hbar$.

Hence $(\mathscr{S}, *_\hbar)$ can be naturally embedded in $\mathscr{T}_\tau / \mathscr{R}_\tau$. If $a *_\hbar b = c$, then this implies that $a \otimes b = c \bmod \mathscr{R}_\tau$ in the space $\mathscr{T}_\tau$. We also note that the product $*_\hbar$ is continuous in the topology τ.

Remark that for every $\hbar \neq 0$, we see that

$$e_{\odot}^{\frac{t}{\hbar} X_1 \odot X_2} = \sum_k \frac{1}{k!} (\frac{t}{\hbar})^k X_1^k \odot X_2^k$$

is an element of $\mathscr{S}_\tau$ for every $0 < \tau < \frac{1}{2}$ and similarly,

$$X_2^\circ = \frac{1}{X_2} (1 - e_{\odot}^{\frac{2i}{\hbar} X_1 \odot X_2}) \in \mathscr{S}_\tau, \quad 0 < \tau < \frac{1}{2},$$

where the right-hand side stands for

$$\sum_{k=1}^{\infty} \frac{1}{k!} (\frac{2i}{\hbar})^k X_1^k \odot X_2^{k-1}.$$

Since $X_i \in \mathscr{S}_{\tau'}$, $\frac{1}{2} < \tau'$, $X_2 *_\hbar X_2^\circ$, $X_2^\circ *_\hbar X_2$ are well-defined, a direct calculation using the Moyal product formula gives that $X_2 *_\hbar X_2^\circ = 1$, but $X_2^\circ *_\hbar X_2 = 1 - 2 e_{\odot}^{\frac{2i}{\hbar} X_1 \odot X_2}$.

Considering a polynomial approximation of X_2° in $\mathscr{T}_\tau$ and using the continuity of the left multiplication $X_2 *_\hbar$ in $\mathscr{S}_\tau$, we see that $X_2 \otimes X_2^\circ = 1 \mod \mathscr{R}_\tau$.

Taking the complex conjugate, we have $\overline{X_2^\circ} \otimes X_2 = 1 \mod \mathscr{R}_\tau$. Since X_2° and $\overline{X_2^\circ}$ are a right-inverse and a left-inverse of X_2 respectively, we have $X_2^\circ - \overline{X_2^\circ} \in \mathscr{R}_\tau$. Hence, we have

$$\frac{1}{X_2} \sin_\odot (\frac{2}{\hbar} X_1 \odot X_2) \in \mathscr{R}_\tau.$$

Since

$$X_2 *_\hbar (\frac{1}{X_2} \sin_\odot \frac{2}{\hbar} X_1 \odot X_2) = \sin_\odot \frac{2}{\hbar} X_1 \odot X_2 + i \cos_\odot \frac{2}{\hbar} X_1 \odot X_2,$$

we have

$$X_2 \otimes (\frac{1}{X_2} \sin_\odot \frac{2}{\hbar} X_1 \odot X_2) - (\sin_\odot \frac{2}{\hbar} X_1 \odot X_2 + i \cos_\odot \frac{2}{\hbar} X_1 \odot X_2) \in \mathscr{R}_\tau.$$

Since the first term is in $\mathscr{R}_\tau$ and $\mathscr{R}_\tau$ is closed under complex conjugation, we see that $\sin_\odot \frac{2}{\hbar} X_1 \odot X_2$, $\cos_\odot \frac{2}{\hbar} X_1 \odot X_2 \in \mathscr{R}_\tau$. Hence we have $e_\odot^{\pm \frac{2i}{\hbar} X_1 \odot X_2} \in \mathscr{R}_\tau$.

Then we have

Lemma 5.2 $1 = e_\odot^{\frac{2i}{\hbar} X_1 \odot X_2} \odot e_\odot^{-\frac{2i}{\hbar} X_1 \odot X_2} \in \mathscr{R}_\tau$.

Proof. We first remark that

$$(X_1^m \odot X_2^n) \otimes (\partial_{X_1}^k \partial_{X_2}^l e_\odot^{\pm \frac{2i}{\hbar} X_1 \odot X_2}) \otimes (X_1^{m'} \odot X_2^{n'})$$

is an element of $\mathscr{R}_\tau$. This is because $\partial_{X_i} f$ can be written by using the commutator bracket.

Hence $(X_1^m \odot X_2^n) \odot e_\odot^{-\frac{2i}{\hbar} X_1 \odot X_2} \in \mathscr{R}_\tau$. This implies

$$(\sum_{k=0}^{n} \frac{1}{k!} X_1^k \odot X_2^k) \odot e_\odot^{-\frac{2i}{\hbar} X_1 \odot X_2} \in \mathscr{R}_\tau.$$

Taking the limit of the above quantity as $n \to \infty$, we see that $1 \in \mathscr{R}_\tau$, under the assumption that $\mathscr{T}_\tau / \mathscr{R}_\tau$ is not a trivial algebra. This is a contradiction, and hence we see that $\mathscr{R}_\tau \not\ni 1$. ■

6. $\mathscr{E}_p(\mathbb{C}^2)$ and $\mathscr{S}_\tau$

We first identify a polynomial on $\mathbb{C}^2$ and an element of the symmetric algebra $\mathscr{S}$ by

$$x^k y^l \longleftrightarrow X_1^k \odot X_2^l.$$

Proposition 6.1 *For every $p > 0$, $\mathscr{S}_{\frac{1}{p}}$ is linearly isomorphic to $\mathscr{E}_p(\mathbb{C}^2)$.*

Proof. Though the well-known estimates for entire functions gives the proposition [3], we will repeat the proof. First, we remark the following relation between semi-norms of polynomials and the symmetric algebras:

$$\|x^ky^l\|_{p,s} \leq \|X_1^k{\circ}X_2^l\|_{\frac{1}{p},(sp)^{-1}} \leq \|x^ky^l\|_{p,sp\delta} \quad \text{for every} \quad s >> 0, \tag{6.1}$$

where $\delta = e^{-(1+\frac{p}{2})}$. To prove these inequalities, we remark that the maximum point of

$$|x^ky^l|e^{-s|\xi|^p}, \quad |\xi|^2 = |x|^2+|y|^2$$

satisfies $k+l = sp|\xi|^p$ and the maximum value is

$$k^{\frac{k}{2}}l^{\frac{l}{2}}e^{-\frac{1}{p}(k+l)}(sp)^{-\frac{1}{p}(k+l)}(k+l)^{(\frac{1}{p}-\frac{1}{2})(k+l)}.$$

It follows that

$$\|x^ky^l\|_{p,s} \leq (sp)^{-\frac{1}{p}(k+l)}(k+l)^{\frac{1}{p}(k+l)}$$

and hence $\|x^ky^l\|_{p,s} \leq \|X_1^k{\circ}X_2^l\|_{\frac{1}{p},(sp)^{-1}}$.

To obtain the next inequality, we remark that $2 < e$ and recall the inequality (2.3). Then, we have

$$(sp)^{-\frac{1}{p}(k+l)}(k+l)^{\frac{1}{p}(k+l)} \leq k^{\frac{k}{2}}l^{\frac{l}{2}}e^{-\frac{1}{p}(k+l)}(sp\delta)^{-\frac{1}{p}(k+l)}(k+l)^{(\frac{1}{p}-\frac{1}{2})(k+l)},$$

where $\delta = e^{-(1+\frac{p}{2})}$. It follows that $\|X_1^k{\circ}X_2^l\|_{\frac{1}{p},(sp)^{-1}} \leq \|x^ky^l\|_{p,sp\delta}$.

The inequality (6.1) naturally yields the embedding

$$\mathscr{S}_{\frac{1}{p}} \hookrightarrow \mathscr{E}_p(\mathbb{C}^2),$$

since $T = \sum a_{k,l}X_1^k{\circ}X_2^l$ and the corresponding $t = \sum a_{k,l}x^ky^l$ satisfy the inequality

$$\|t\|_{p,s} \leq \|T\|_{\frac{1}{p},(sp)^{-1}}.$$

Next, we show that $\mathscr{E}_p(\mathbb{C}^2) \subset \mathscr{S}_{\frac{1}{p}}$.

Since every $f \in \mathscr{E}_p(\mathbb{C}^2)$ is an entire function, f is expressed as a convergent power series $f = \sum c_{m,n}x^my^n$. Let $M(r) = \max_{|\xi|=r}|f(x,y)|$. By Cauchy's integration formula, there is a constant $C_\delta > 0$ such that

$$|c_{m,n}| \leq \frac{M(r)}{r^{m+n}} \leq C_\delta\frac{e^{\delta r^p}}{r^{m+n}}$$

holds for every $\delta > 0$, since every $f \in \mathscr{E}_p(\mathbb{C}^2)$ satisfies $|f(x,y)| \leq Ce^{\delta|\xi|^p}$.

Taking the minimal value of the right-hand side with respect to r for a fixed δ, we have

$$|c_{m,n}| \leq C_\delta(\delta ep)^{\frac{1}{p}(m+n)}(\frac{1}{m+n})^{\frac{1}{p}(m+n)}. \tag{6.2}$$

Using (6.2), we easily see that $f \in \mathscr{S}_{\frac{1}{p}}$, if δ is sufficiently small. This gives also that every $f \in \mathscr{E}_p(\mathbb{C}^2)$ is also an element of $\mathscr{S}_{\frac{1}{p}}$. Hence we see that $\mathscr{E}_p(\mathbb{C}^2)$ and $\mathscr{S}_{\frac{1}{p}}$ are linearly isomorphic. ■

6.1. Proof of Theorem 1.2.

Theorem 3.3 gives that if $p \leq 2$, then $(\mathscr{E}_p(\mathbb{C}^2); *_\hbar)$ is a Fréchet algebra. Theorem 3.5 gives (ii) of Theorem 1.2.

Hence, to complete the proof of Theorem 1.2, it is sufficient to show that $(\mathscr{E}_p(\mathbb{C}^2); *_\hbar)$ is a deformation quantization of $(\mathscr{E}_p(\mathbb{C}^2), \cdot, \{\ \})$.

Since ∂_u, ∂_v are continuous linear operators on $\mathscr{E}_p(\mathbb{C}^2)$, $(\mathscr{E}_p(\mathbb{C}^2), \cdot, \{\ \})$ is a Fréchet-Poisson algebra. By Theorem 3.3, the concrete formula (1.2) is absolutely convergent in $\hbar$. Thus, $f *_\hbar g$ is analytic in $\hbar$ for any $f, g \in \mathscr{E}_p(\mathbb{C}^2)$.

7. The product formula for e^{isxy}

As mentioned in the introduction, y has a right-inverse y° and a left-inverse $y^\bullet$ at the same time in the space $\mathscr{E}_p(\mathbb{C}^2)$ for $p > 2$. Hence the associativity of the product $*_\hbar$ fails if $\hbar > 0$:

$$y^\bullet *_\hbar (y *_\hbar y^\circ) \neq (y^\bullet *_\hbar y) *_\hbar y^\circ.$$

In this section, we show that if $p > 2$, and $\hbar > 0$, then $\mathscr{E}_p(\mathbb{C}^2)$ is not closed under the product $*_\hbar$. In fact, we see that $y^\bullet *_\hbar y^\circ$ diverges. See also [6] for details.

Since xy is a polynomial, the product $(ixy) *_\hbar f(x,y)$ is well-defined for every $\mathscr{E}_p(\mathbb{C}^2)$. Thus, we consider a differential equation

$$\frac{d}{dt} f_t = (ixy) *_\hbar f_t(x,y), \quad f_0(x,y) = f(x,y).$$

By the Moyal product formula (1.2), this is written as the ordinary evolution equation

$$\frac{d}{dt} f_t = (ixy) f_t - \frac{\hbar}{2}(x\partial_x f_t - y\partial_y f_t) + i\frac{\hbar^2}{4}\partial_y\partial_x f_t.$$

The existence of the solution does not hold in general, but the uniqueness holds in the real analytic category in t. Hence if the initial function f_0 is a function $F(xy)$ of xy, we may suppose that the solution is also a function of xy.

Under this assumption, we consider the differential equation for $z = xy$:

$$\frac{d}{dt} f_t(z) = izf_t(z) + i\frac{\hbar^2}{4}(f_t'(z) + zf_t''(z)). \tag{7.1}$$

The solution with initial condition 1 may be written as the $*_\hbar$-exponential function e_*^{itz}, which is given in fact by

$$e_*^{itz} = \frac{1}{\cosh \frac{\hbar t}{2}} e^{i(\frac{2}{\hbar}\tanh\frac{\hbar t}{2})z}, \quad t \in \mathbb{C} - \{(m + \tfrac{1}{2})\frac{i}{\hbar};\ m \in \mathbb{Z}\}. \tag{7.2}$$

The exponential law $e_*^{isz} *_\hbar e_*^{itz} = e_*^{i(s+t)z}$ holds by uniqueness whenever $s, t, s+t \in \mathbb{C} - \{(m+\frac{1}{2})\frac{i}{\hbar};\ m \in \mathbb{Z}\}$.

Rewriting the exponential law, we obtain the product formula

$$ae^{isz} *_\hbar be^{itz} = \frac{4ab}{4+\hbar^2 st} e^{\frac{4i(s+t)}{4+\hbar^2 st}z}. \tag{7.3}$$

It is easy to check the associativity when products are defined, and the inverse of ae^{isz} is $\frac{a^{-1}}{4}(4-\hbar^2 s^2)e^{-isz}$ for $4-\hbar^2 s^2 \neq 0$. If $s = \pm\frac{2}{\hbar}$, then $e^{\pm\frac{2i}{\hbar}z}$ are not invertible but have the idempotent property:

$$2e^{\pm\frac{2i}{\hbar}z} *_\hbar 2e^{\pm\frac{2i}{\hbar}z} = 2e^{\pm\frac{2i}{\hbar}z}. \tag{7.4}$$

However, $e^{\frac{2i}{\hbar}z} *_\hbar e^{-\frac{2i}{\hbar}z}$ is not defined. We refer to elements $2e^{\pm\frac{2i}{\hbar}z}$ as *vacuums*.

By the above argument, we see that

$$e_*^{itxy} = \frac{1}{\cosh\frac{\hbar t}{2}} e^{(\frac{2i}{\hbar}\tanh\frac{\hbar t}{2})xy}$$

is an element of $\mathscr{E}_p(\mathbb{C}^2)$, $p > 2$, for $t \in \mathbb{R}$. If $\hbar$ is a fixed positive real, then

$$\int_0^\infty e_*^{itxy}\,dt, \qquad \int_{-\infty}^0 e_*^{itxy}\,dt$$

are also elements of $\mathscr{E}^2(\mathbb{C}^2)$, $p > 2$ and $-i\int_0^\infty e_*^{itxy}\,dt$, and $i\int_{-\infty}^0 e_*^{itxy}\,dt$ are two different inverse elements of xy, that is

$$(xy) *_\hbar (-i\int_0^\infty e_*^{itxy}\,dt) = (xy) *_\hbar (i\int_{-\infty}^0 e_*^{itxy}\,dt) = 1.$$

Thus the associativity fails again: By denoting

$$(xy)_{0i+}^{-1} = -i\int_0^\infty e_*^{itxy}\,dt, \quad (xy)_{0i-}^{-1} = (i\int_{-\infty}^0 e_*^{itxy}\,dt)$$

we have

$$((xy)_{0i+}^{-1} *_\hbar (xy)) *_\hbar (xy)_{0i-}^{-1} \neq (xy)_{0i+}^{-1} *_\hbar ((xy) *_\hbar (xy)_{0i-}^{-1}).$$

This is because

$$(xy)_{0i+}^{-1} - (xy)_{0i-}^{-1} = -i\int_{-\infty}^\infty e_*^{itxy}\,dt \neq 0.$$

The right-hand side may be written as $-i\delta_*(xy)$ and the above identity corresponds to a relation of hyperfunctions [4].

If $\hbar > 0$, then by the well-known Hansen-Bessel formula (cf. [2], p. 1802), we see that

$$\int_{-\infty}^\infty e_*^{itxy}\,dt = \int_{-\infty}^\infty \frac{1}{\cosh\frac{\hbar t}{2}} e^{(\frac{2i}{\hbar}\tanh\frac{\hbar t}{2})xy}\,dt = \frac{\pi}{2}J_0(\frac{2}{\hbar}xy).$$

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On the deformation aspects of the replica thermalisation of the $SU(2)$-invariant Thirring model

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Abstract. We introduce an S-matrix preserving replica-thermalisation of integrable massive quantum field theories in $1+1$-dimensions within the context of form factors. The deformation character of the methods used therein is being highlighted. We then solve a deformed version of the $SU(2)$-invariant Thirring model employing hypergeometric solutions of the rational sl_2-type quantum Knizhnik-Zamolodchikov equation at generic level and multiperiodic Barnes functions.

Keywords: form factors, replica, deformation, qKZ-equations

Mathematics Subject Classifications (2000): 81R50, 33C60

1. Introduction

It has been mentioned by many speakers during the conference that a *leitmotif* in the scientific work of Moshé Flato is the concept of deformation. He has been advocating that in physics one should look whether a well established structure admits a deformation. One of the most important applications of this idea is of course *deformation quantisation* [1], the fundamental insight being that we actually can understand quantum mechanics as a deformation of the algebraic setup of classical mechanics. The deformation parameter is nothing else than Planck's constant $\hbar$. Once we have accepted this framework (or the category in which the deformation is carried out, to be precise) one could try to deform quantum mechanics even further. However, there are severe cohomological obstructions prohibiting to do this. This means that quantum mechanics is stable with respect to deformations. I am grateful for many enlightening discussions with Moshé on this topic resulting in the work [2], where this rigidity has been established and applied to a certain structure arising in the theory of quantum groups.

Very much in the same spirit this kind of logic appears also in another of Moshé's important contributions, which is the theory of singletons in anti-de Sitter space, see [3]. Here the starting point is Wigner's classification irreducible representations of the Poincaré group and their identification with particle states. One, however, realises that the Poincaré group can be obtained from the anti-de Sitter group by group contraction and therefore the former can be understood as a deformation of the latter, which is stable in the aforementioned sense. One could then ask whether the irreducible representations of the anti-de Sitter group could also play a fundamental role in physics. In fact, this has been established in many

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circumstances as can be read off a recent article [3] and the list of literature cited therein, and also by the results in the context of M-theory.

In this contribution we would like to argue that a deformation absolutely parallel in spirit to the ones described above, does appear within the context of massive integrable field theories in $1+1$-dimensional space-time. First we are going to describe the framework or the category in which we deform a given quantum field theory (QFT). Then we show that this deformed theory can actually be solved exactly using the deformed (quantum) Knizhnik-Zamolodchikov (qKZ) equations.

Before starting off we introduce the $SU(2)$-invariant Thirring model (ITM) which will provide the playground in this paper. The details may be found e.g. in [4]. We should, however, say that the general results to be mentioned below are not restricted to this particular model. For the sake of orientation let us display the classical Lagrangian. It is given in terms of a two-component spinor ψ, valued in the fundamental representation of $SU(2)$.

$$\mathscr{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - g(\bar{\psi}\sigma^{a}\gamma^{\mu}\psi)(\bar{\psi}\sigma_{a}\gamma_{\mu}\psi). \tag{1}$$

Here σ^a are the Pauli matrices. The QFT associated with this model exhibits dynamical mass generation and is believed to be asymptotically free. The physical particles of the theory are an SU(2) doublet of massive kinks. The index $\varepsilon \in \{\pm\}$ refers to either of these kinks. Moreover it is known that the model is integrable, and the factorised S-matrix is given as a function of the kink-rapidity z by the following expression:

$$S_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}(z) = S_0(z)\frac{z\ \delta_{\varepsilon_1}^{\varepsilon_1'}\delta_{\varepsilon_2}^{\varepsilon_2'} - i\pi\ \delta_{\varepsilon_1}^{\varepsilon_2'}\delta_{\varepsilon_2}^{\varepsilon_1'}}{z - i\pi}, \tag{2}$$

where the scalar part S_0 of the S-matrix is

$$S_0(z) = \frac{\Gamma\left(\frac{1}{2} + \frac{z}{2i\pi}\right)\Gamma\left(-\frac{z}{2i\pi}\right)}{\Gamma\left(\frac{1}{2} - \frac{z}{2i\pi}\right)\Gamma\left(\frac{z}{2i\pi}\right)}. \tag{3}$$

We mention that without S_0 the expression in (2) coincides with the rational R-matrix of the quantum group sl_2 in the defining representation.

2. Deformation-thermalisation and the form factor approach

The deformation we are going to describe here is actually associated to a well-known thermalisation phenomenon in QFT. We consider the Rindler wedge W in a $1+1$-dimensional Minkowski space, with K being the Lorentz boost generator. Let $\mathscr{O}_i$ for $i = 1, \ldots, n$ be local operators with support inside W. The Bisognano-Wichmann theorem [5] then states that the vacuum of a Minkowski space QFT inside W looks like a thermal state of inverse temperature $\beta = 2\pi$ with respect to the Killing time in W. This temperature is both physically and mathematically linked to the Unruh temperature $T_U = 1/2\pi$, which is seemingly of a very different origin. Symbolically we can encode this fact into

$$\langle 0|\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|0\rangle \sim \mathrm{Tr}[\exp(2\pi K)\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)]\,. \tag{4}$$

Note that we use the symbol Tr only as a mnemonic to remind of the thermal structure of the matrix element. Actually for a continuum QFT the object we are dealing with is meaningful only within the framework of modular theory and our operators $\mathcal{O}_i$ will in general not be of trace-class and moreover the non-compact nature of K would spoil the naive game anyway. A detailed treatment of this, however, would be beyond the scope of this paper.

Now we bring the deformation idea into play. Obviously the value $\beta = 2\pi$ is very special in our physical system. Even though one mainly deals with this particular value in standard QFT we will not be in a stable context or in the most general setup would we not allow arbitrary values for β. However, we would like to constrain the category we are working with a bit. Namely we want to vary the value of the "inverse temperature" β which amounts to replace

$$\exp(2\pi K) \to (\exp(2\pi K))^{\beta/2\pi} = \exp(\beta K), \tag{5}$$

while leaving the factorised scattering property, hence in particular the S-matrix as in (2), of the model under consideration unaffected. This kind of situation arises also in the broader context of quantum gravity. The procedure outlined can be considered as a concrete realisation of the *replica* idea of Callan and Wilczek [6].

What has been said so far in this section remains an empty application of the deformation concept if we were not able to implement our idea into a realistic QFT. Or in concrete terms the question is how we can define correlation functions of local operators consistent with the replica idea (5) and avoiding mathematical ambiguities.

The solution of this problem was given by Niedermaier in a series of papers [7]. He showed that the replica deformation outlined before can be consistently implemented on the level of form factors [4]. A form factor is a matrix element of a local operator $\mathcal{O}(x)$ between the vacuum and an n-particle scattering state. These states depend on rapidities z_i and are created from the vacuum by means of Faddeev-Zamolodchikov operators $A_{a_i}(z_i)$, where the index a_i refers to the charge of the state. Encoding this in a formula means

$$\begin{aligned} F_{a_1\ldots a_n}(z_1,\ldots,z_n) &= \\ &\langle 0|\mathcal{O}(x)|z_1,\ldots,z_n\rangle_{a_1\ldots a_n} \sim \mathrm{Tr}\left[e^{2\pi K}\mathcal{O}(x)A_{a_1}(z_1)\cdots A_{a_n}(z_n)\right], \end{aligned} \tag{6}$$

where we are still using the trace mnemonic described before. Employing the standard Källén-Lehmann spectral series this form factor enables us in principle to compute the Wightman functions of the QFT by the following expression (omitting for the moment the charge indices of the form factor).

$$\langle 0|\mathcal{O}_1(x)\mathcal{O}_2(0)|0\rangle = \sum_n \frac{1}{n!}\int F^{\mathcal{O}_1}(z_1,\ldots,z_n)(F^{\mathcal{O}_2}(z_1,\ldots,z_n))^* e^{-m|x|\sum_{i=1}^n \cosh z_i}\prod_{i=1}^n \frac{\mathrm{dz_i}}{2\pi}. \tag{7}$$

One of the most fascinating aspects of integrable massive quantum field theories in $1+1$-dimensions is that one can in principle compute the form factors (6) exactly. In concrete terms this means that there exists a set of equations which enables to compute the expression (6) in a non-perturbative way. This was first realised in [8] but essentially carried out by Smirnov [4] for $\beta = 2\pi$. It was also Smirnov who proved that once one has these equations at hand, they do imply locality of the QFT.

It is one of the achievements of Niedermaier [7] that he could prove the results of Smirnov within the context of axiomatic QFT using modular theory and thereby generalising the setup to the deformed framework outlined above.

The first two conditions the form factor has to satisfy are the following

$$\begin{aligned} &F_{a_1,a_2\ldots a_n}(z_1+i\beta,z_2,\ldots,z_n) = \eta\, F_{a_2\ldots a_n,a_1}(z_2,\ldots,z_n,z_1),\\ &F_{a_1,\ldots a_n,a_{n-1}}(z_1,\ldots,z_n,z_{n-1}) = S_{a_{n-1}a_n}^{a'_{n-1}a'_n}(z_{n-1}-z_n)F_{a_1,\ldots,a'_{n-1}a'_n}(z_1,\ldots,z_{n-1},z_n). \end{aligned} \tag{8}$$

The parameter η is a pure phase, its actual value depends on the particular operator $\mathcal{O}$ described by the form factor.

It is interesting to mention already at this stage that because the S-matrix entering (8) in our case corresponds to the rational R-matrix of the quantum group sl_2 it can be shown that Equations (8) imply a quantum Knizhnik-Zamolodchikov equation. From this point of view one can realise another aspect of our β-deformation. If $\beta = 2\pi$ we are in the particular case of a qKZ equation at level zero [9]. This is exactly the special case studied by Smirnov [4]. Hence, it is absolutely natural from the qKZ point of view to ask whether there is a physical interpretation of the qKZ-solutions at generic level, which means for $\beta \neq 2\pi$. However, the physics of the model in question requires further properties of the form factor associated with its analytic structure. Let $C = i\sigma^2$ be the charge conjugation matrix of the ITM. We then get the following residue equations for the form factor

$$\begin{aligned} \operatorname*{res}_{z_n=z_{n-1}+i\pi} F_{a_1,\ldots a_{n-1},a_n}(z_1,\ldots,z_{n-1},z_n) &= i\,C_{a_{n-1}a_n}F_{a_1,\ldots a_{n-2}}(z_1,\ldots,z_{n-2}),\\ \operatorname*{res}_{z_n=z_{n-1}-i\pi} C^{a_{n-1}a_n}F_{a_1,\ldots a_{n-1},a_n}(z_1,\ldots,z_{n-1},z_n) &= i\,F_{a_1,\ldots a_{n-2}}(z_1,\ldots,z_{n-2}). \end{aligned} \tag{9}$$

Note that the residue equation for $z_n = z_{n-1} + i\pi$ is similar to the condition satisfied by traces of vertex operators in lattice models [10]. Without the second equation (9) however the recursion $n-2 \to n$ would be highly ambiguous. Taking advantage of the qKZ equation (8) to implement the analytic continuation $z_n \to z_n + i\beta$, the second equation (9) can be rewritten as follows

$$\begin{aligned} \operatorname{res}_{z_n=z_{n-1}-i\pi+i\beta} F_{a_1\ldots a_n}(z_1,\ldots,z_{n-1},z_n) \;&=\; \eta F_{b_1\ldots b_{n-2}}(z_1,\ldots,z_{n-2})\times \\ &\times C_{a_n c} S_{c_1 a_1}^{c\, b_1}(z_{n-1}-z_1)\cdots S_{a_{n-1}a_{n-2}}^{c_{n-1}b_{n-2}}(z_{n-1}-z_{n-2}). \end{aligned} \tag{10}$$

If we are in the deformed context $\beta \neq 2\pi$ is is easy to see that the form factor (6) can have a very rich pole structure in the region $|Im(z)| < 2\pi$. In contrast to that, if $\beta = 2\pi$ then period of analytic continuation is 2π and it can be shown that (9) and (10) combine to give the standard kinematical residue equations introduced by Smirnov [4].

To summarise this section we have shown that it is possible to deform an integrable massive QFT according to the prescription (5) preserving the S-matrix of the model. This deformation can be carried out in the context of the form factor approach and in fact one can find non-perturbative solutions of the QFT in the deformed region via (8), (9), and (10). From the point of view of the qKZ equations it seems absolutely natural to have form factors at $\beta = 2\pi$. We should, however, stress that the physics in the deformed model shows essentially new features. For example the structure of the conserved charges will be different. Then physically notable is the expected change in the kinematics, which for instance causes an increase of mass eigenvalues in the asymptotic states, or there may be circumstances where not the entire phase space is accessible by the model. These and other interesting features are discussed in [7].

3. Minimal form factors and Barnes' multi-periodic functions

We are now going to support the argument that it is mathematically absolutely natural to have the deformation parameter $\beta \neq 2\pi$. For that we consider the equations (8) in the case of only two particles present (so called minimal form factor equations). This problem essentially reduces to the following equations involving only the scalar part of the S-matrix (3).

$$F^m(z) = S_0(z)F^m(-z), \qquad F^m(z+i\beta) = F^m(-z). \tag{11}$$

The standard solution procedure of this equation is that one brings $S_0(z)$ into an integral representation and then solves (11). This has been done in [4] for the undeformed case of $\beta = 2\pi$. However, as was shown in [11] for a whole class of integrable models, this approach does not reveal the mathematical beauty of the solution spaces of (11). Upon assuming certain minimality conditions on F^m a natural solution can be found in terms of functions which were introduced by Barnes about one hundred years ago, see [11]. In our case we use Barnes' diperiodic Γ-function. This object is defined by a contour integral of Hankel type, where γ is the usual Euler constant. The periods are ω_1 and ω_2, and do enter symmetrically into the definition.

$$\log\Gamma_2(x|\omega_1,\omega_2) = \frac{1}{2\pi i}\int_{C_H} \frac{\exp(-xt)(\log(-t)+\gamma)}{\prod_{i=1}^{2}(1-\exp(-\omega_i t))}\frac{\mathrm{d}t}{t}. \tag{12}$$

The most important property of this function is its periodicity in the sense of Barnes, which means that

$$\frac{\Gamma_2(x+\omega_1|\omega_1,\omega_2)}{\Gamma_2(x|\omega_1,\omega_2)} \sim \frac{1}{\Gamma(x|\omega_2)}. \tag{13}$$

It is now straightforward to check that

$$F^m(\theta) = \frac{\Gamma_2(\pi - i\theta|2\pi,\beta)\Gamma_2(\pi+\beta+i\theta|2\pi,\beta)}{\Gamma_2(-i\theta|2\pi,\beta)\Gamma_2(\beta+i\theta|2\pi,\beta)} \tag{14}$$

is a solution to (11) for generic β. And as we have two periods in (14) we realise at once that it is natural to have one period different from 2π, while the undeformed case studied in [4] is just the degeneration of (14) for coinciding periods $\omega_1 = \omega_2 = 2\pi$. Even though we stick to the ITM in this note, it is worth emphazising that we get equivalent results for other integrable models as well, see [11].

4. General solution and the qKZ equation

Up to now we have been explaining the naturalness of having a deformation parameter $\beta \neq 2\pi$ in the form factor approach. In order to make really sense of this we have to solve Equations (8)–(10) for an arbitrary kink configuration in the ITM.

I guess it would be in Moshé's spirit to say that once we have a mathematically firm fundament the rest is just computational technique. It turns out that this is true in the present case as we can take advantage of the classification of the solutions of the sl_2-qKZ equation with rational R-matrix carried out by Tarasov and Varchenko [12]. The details on how one can construct the form factor solutions from the qKZ ones can be found in [13], we will here merely sketch the results in order to indicate how the system works.

In the ITM we are concerned with kinks and anti-kinks which correspond to the fundamental representation V of $SU(2)$. Hence an n-particle state will be an element in $V^{\otimes n}$. Define the Pauli matrices acting in this tensor product space as $\Sigma^a = \sum_{i=1}^n \sigma^a$. The weight subspaces of $V^{\otimes n}$ are then given by:

$$(V^{\otimes n})_l = \{v \in V^{\otimes n} | \Sigma^3 v = (n-2l)v\}, \tag{15}$$

where we can interpret the quantum number $j = n/2 - l$ as the isospin carried by a local operator which is described by the form factor (6). Due to the intertwining properties of the local operator we can restrict attention to yet another subspace of $(V^{\otimes n})_l$, which is the subspace of singular vectors $(V^{\otimes n})_l^{sing}$ whose elements are annihilated by Σ^+. We will consider only those solutions of (8) which are valued in $(V^{\otimes n})_l^{sing}$. Following [9] we will use the following convention to label the vectors in this space. Let $M = \{m_1 < m_2 < \ldots < m_l\} \subseteq \{1,2,\ldots,n\}$, such that the number of elements of M is $\#M = l$. In other words this set labels the positions at which the form factor (6) has kinks of type "−" in the scattering state, i.e., $M = \{i|\varepsilon_i = -\}$. Hence, we can write $v_M := v_{\varepsilon_1} \otimes \ldots \otimes v_{\varepsilon_n}$.

This in turn means that for the indices at the form factor (6) we can make the identification $F_{a_1,\dots,a_n}(z_1,\dots,z_n) = F_M(z_1,\dots,z_n)$.

The solutions of the rational qKZ equation in the case of tensor products of fundamental representations of sl_2 can be found in [12]. According to [13] they can be rewritten along the lines indicated in [9] in the following way:

$$\begin{aligned}\mathscr{I}^P_M(z_1,\dots,z_n) \;=\; &\sum_{\#M=l}\int_{C_1}\cdots\int_{C_l} w_M(t_1,\dots,t_l)P(\xi_1,\dots,\xi_l;\zeta_1,\dots,\zeta_n)\times \\ &\times\prod_{a=1}^{l}\prod_{j=1}^{n}\Phi(t_a-z_j)\prod_{1\le a<b\le l}\psi(t_a-t_b)\,\mathrm{d}t_1\cdots\mathrm{d}t_l\; v_M.\end{aligned} \tag{16}$$

We shall briefly explain the entities in this hypergeometric integral, which is a solution to the qKZ equation for the representation considered in this paper. The structure of the integration contours C_l is not really important in this paper and we refer to [13]. The analytic information on the kink-state involved is encoded in the function

$$w_M(t_1,\dots,t_l) = \mathrm{Asym}\left(\prod_{a=1}^{l}\left(\frac{1}{t_a - z_{m_a} - 3i\pi/2}\prod_{1\le k<m_a}\frac{t_a - z_k - i\pi/2}{t_a - z_k - 3i\pi/2}\right)\right), \tag{17}$$

where the antisymmetrisation has to be performed over all variables $t_1,\dots,t_l$. The following objects have a deep meaning in the geometry of hypergeometric functions [12]. We have

$$\Phi(y) := \Gamma\left(\frac{\pi/2+iy}{\beta}\right)\Gamma\left(\frac{\beta-3\pi/2-iy}{\beta}\right)\exp(\frac{\pi}{\beta}y), \tag{18}$$

and

$$\psi(y) := \frac{1}{\pi^3\beta}\Gamma\left(\frac{\pi+iy}{\beta}\right)\Gamma\left(\frac{\pi-iy}{\beta}\right)\sinh(\frac{\pi}{\beta}\,y). \tag{19}$$

Last but not least $P(\xi_1,\dots,\xi_l;\zeta_1,\dots,\zeta_n)$ is a polynomial in variables $\xi_a = \exp(-2\pi t_a/\beta)$ with complex coefficients and certain requirements on the partial degrees in the variables ξ, see [12, 13]; note that $\zeta_k = \exp(-2\pi z_k/\beta)$. The object P is yet undetermined as $\mathscr{I}^P_M$ is a solution of the qKZ equation independent of the actual form of P. In fact, P will serve to classify the local operator content of the QFT.

Using the results of the previous section, we can now prove that

$$F_M(z_1,\dots,z_n) = \frac{c_l}{(2\pi i)^l}\prod_{n\ge i>j\ge 1}F^m(z_i-z_j)\times\mathscr{I}^P_M(z_1,\dots,z_n), \tag{20}$$

satisfies the first bit (8) of the deformed form factor equations for some constant c_l. We would like to stress again, that the limiting value $\beta = 2\pi$ is naturally within the solutions (20), moreover from the viewpoint of qKZ solution spaces it would even be unnatural to restrict to the single value $\beta = 2\pi$.

5. Kinematical residue equations and the Noether currents

The harder part in proving that (20) really is a consistent form factor, is to establish conditions on the free parameters P arising from the kinematical residue equations (9) or (10). To evaluate the residues of (20) one has to carefully consider the analytic structure (poles and zeroes) of the hypergeometric integral (16). This structure has an important geometrical meaning [12] and is among the most beautiful features of the qKZ solutions.

The details of the calculation can be found in [13] and we will merely state one of the conditions on $P(\xi_1,\dots,\xi_l;z_1,\dots,z_n)=P_{l,n}$ in order to get a flavour of what it looks like.

$$
\begin{aligned}
P_{l,n}\Big|_{\substack{t_l=z_{n-1}+3i\pi/2\\ z_n=z_{n-1}+i\pi}} \times \prod_{a=1}^{l-1} \frac{(-1)\,e^{2\pi(t_a-z_{n-1}-i\pi/2)/\beta}}{\sinh\frac{\pi}{\beta}(t_a-z_{n-1}-i\pi/2)\sinh\frac{\pi}{\beta}(t_a-z_{n-1}-5i\pi/2)} \times \\
\times \prod_{j=1}^{n-2} e^{\pi(z_{n-1}-z_j+3i\pi/2)/\beta} \qquad \sim \qquad P_{l-1,n-2}.
\end{aligned}
\tag{21}
$$

The "$\sim$" means that this is an equality up to a phase, which is permitted by the requirements on the form factors outlined before.

The main result obtained in [13] is then that once we find solutions of the recursion equation (21) (and another one of similar type arising from (10)) we have established that there exist infinite sequences of form factors (20), where each sequence is labelled by an invariant isospin $j=n/2-l$ and the solution spaces of (21) classify the local operator content of the QFT associated with the $SU(2)$-invariant Thirring model. The statements just made are true for any value of the deformation parameter β, and only for $\beta=2\pi$ are we in the context of the standard unthermalised Thirring model as it was described by Smirnov [4]. A discussion of the ambiguities of the solution spaces of (21) can be found in [13].

To conclude we would like to write down a solution [13] of (21) which is associated with the Noether current of the ITM for an arbitrary value of the deformation parameter.

Let j^a_μ be the Noether current associated with the Lagrangian (1); μ is the Lorentz index and a the $SU(2)$-index as introduced before. It is convenient to switch to lightcone coordinates both in Minkowski space and in internal space, i.e.,

$$
j^a_\pm = j^a_0 \pm j^a_1, \qquad j^\pm_\sigma = j^1_\sigma \pm i\, j^2_\sigma, \qquad \sigma=\pm. \tag{22}
$$

Then we can introduce an index $\tau=\pm,3$, labelling the components of the lightcone currents as j^τ_σ. A solution of Equations (21) is then up to a constant given by

$$
\begin{aligned}
P^\sigma_{l,n} \sim e^{\pi(2l-n/2+\sigma)\Sigma z_j/\beta} \prod_{a=1}^{l} e^{-2\pi(l+\sigma)t_a/\beta} \times \\
\times \prod_{1\le a<b\le l} \sinh\frac{\pi}{\beta}(t_a-t_b+i\pi)\sinh\frac{\pi}{\beta}(t_a-t_b-i\pi),
\end{aligned}
\tag{23}
$$

It was shown in [13] that the form factors with the insertion of this object correspond to the form factors of the current j_σ^- in (22). The ones of the other currents are obtained from this one by obvious sl_2 action.

This at least gives an existence proof for solutions to (21) and hence we have established that we can exactly solve the replica deformed version of the $SU(2)$-invariant Thirring model using the method of form factors.

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Improved perturbation expansion in quantum chromodynamics

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Abstract. The problem of improving the perturbation expansion in quantum chromodynamics at low energies is considered. Certain modification of the conventional running coupling constant is proposed, which removes the singularity of the effective coupling at nonzero energy. The effect of such modification on the QCD predictions is discussed. It is argued, that such modification may be considered as an unconventional choice of the renormalization scheme.

Keywords: quantum chromodynamics, perturbation expansion, renormalization schemes

Mathematics Subject Classifications (2000): 81T15, 81V05, 41A58

Despite advances in nonperturbative methods the perturbation expansion remains an important tool in practical applications of quantum chromodynamics (QCD). In most cases the so called renormalization group (RG) improved perturbation expansion is used, i.e., an expansion in terms of the running coupling constant (RCC). In the simplest case of a totally inclusive quantity that depends on a single variable Q the RG-improved expression for the QCD correction δ takes the form:

$$\delta(Q) = a(Q)[1 + r_1 a(Q) + r_2 a^2(Q) + \ldots], \tag{1}$$

where $a = g^2/(4\pi^2)$ is the RCC satisfying the equation:

$$Q\frac{da}{dQ} = \beta(a) = -ba^2\,(1 + c_1 a + c_2 a^2 + \ldots). \tag{2}$$

The appearance of RCC is a consequence of partial resummation of the perturbation expansion to all orders. Such a resummation is useful for expressing QCD predictions at high energies, because the RCC is decreasing with increasing energy due to the asymptotic freedom property of QCD. On the other hand, the RCC increases with decreasing energy, and it may even become singular at nonzero value of energy. Obviously, this type of behavior seriously limits the reliability of the RG-improved perturbation expansion in QCD at low energies, which manifests itself for example via substantial renormalization scheme (RS) dependence of finite order predictions. This fact is usually taken as unavoidable feature of QCD and the indication, that at low energies in QCD it is necessary to use nonperturbative methods.

It is of course beyond doubt that QCD requires a nonperturbative treatment in the low energy region. However, the breakdown of the RG-improved perturbation expansion in QCD at low energies may result to some extent from the fact, that at certain energies the conventional RCC ceases to be a useful expansion param-

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eter. Let us note that the infrared behavior of RCC (in particular, the type and the location of the singularity) is determined by the *highest* term retained in the perturbative approximant for the β-function, and thus it strongly depends on the order of the expansion and the choice of the renormalization scheme. It seems therefore worthwhile to investigate other ways to define the effective QCD expansion parameter, which would lead to a better behavior at low energies. It is also important to study, what is exactly the range of energies, for which the modified perturbation expansion may be still meaningfully applied: is it $Q \geq \Lambda$ (where Λ is the QCD scale parameter), or is it perhaps only, say, $Q \geq 5\Lambda$. In many applications this would make an important difference.

A natural way to define a modified effective expansion parameter is to specify a modified form of the β-function. In order to obtain an effective expansion parameter which would be free from the singularity at nonzero value of energy we must introduce a β-function which deviates from the simple polynomial resulting from a truncation of the perturbation expansion. For example, it has been noted already some time ago, that it is possible to avoid singularity in RCC at nonzero energy by introducing exponentially small — i.e., essentially nonperturbative — terms into the β-function [1, 2]. (For other propositions to modify the RCC see [3, 4, 5, 6, 7].) However, in our approach we would like to avoid inserting the essentially nonperturbative effects by hand, and we would like to stay as close as possible to the standard perturbative framework. In fact, as we shall explain later, we would like to exploit in a slightly unconventional way the arbitrariness in the β-function resulting from the freedom of choice of the renormalization scheme.

Our proposition is to replace the polynomial approximant by a nonpolynomial expression which is analytic in some neighborhood of the real positive axis in the complex-a plane. In order to preserve consistency with the conventional perturbation expansion we shall require, that the expansion coefficients of this expression around $a = 0$ should coincide to the considered order with the coefficients in the conventional expression. Additionally, in order to avoid the singularity of the RCC at nonzero energy we shall require, that for large a the modified β-function should typically behave as a^k, where $k \leq 1$. These are of course very general requirements, that by no means specify the modified β-function uniquely. Further constraints on the modification of the β-function would come from the requirement, that the modified evolution of RCC should match closely the behavior emerging from nonperturbative considerations.

It may seem that replacing a polynomial by, say, a rational function we deviate in a substantial and arbitrary way from the original theoretical expression. We want to point out, however, that such a modification may be viewed as a specific choice of the renormalization scheme at a given order of perturbation expansion. As is well known, the β-function depends on the choice of RS. Although the first two expansion coefficients are universal in the class of mass and gauge independent schemes, the coefficient c_2 and all β-function coefficients of higher order are always renormalization scheme dependent. As was pointed out in [8] these

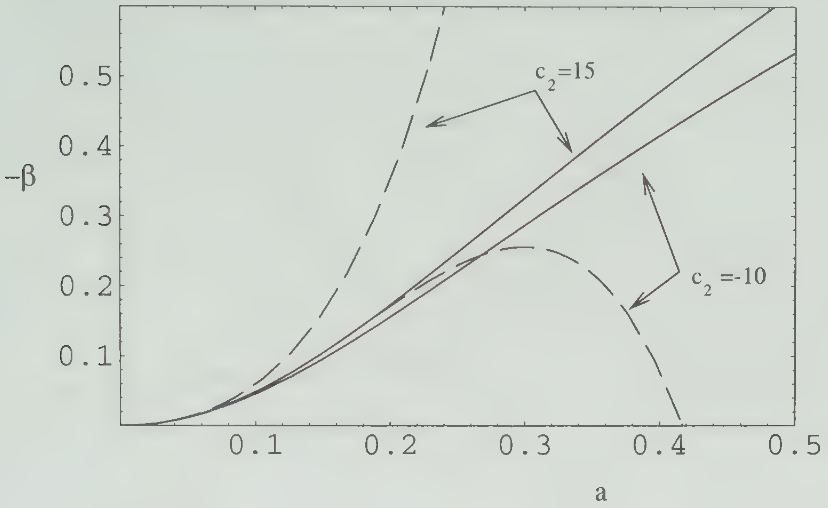

Figure 1. The behavior of the modified NNLO β-function for three "active" quark flavors (solid lines) compared with the conventional NNLO β-function (dashed lines) for $c_2 = 15, -10$ and $\eta = 4$.

coefficients may be even used to *label* the various schemes. In the conventional calculation of the next-to-next-to-leading order (NNLO) predictions one chooses some particular value for c_2 and one assumes $c_k = 0$ for $k \geq 3$. The use of a modified NNLO expression for the β-function corresponds to the situation where the c_2 coefficient is unchanged, but c_k coefficients for $k \geq 3$ take some nonzero value. With our assumption about the analyticity domain the modified β-function is uniquely determined by this infinite sequence of expansion coefficients. The idea to use the β-function with nonzero expansion coefficients at orders higher than the order of approximation for the considered physical quantity is somewhat unconventional, but it is completely consistent with the structure of the perturbation expansion. (Such an idea is also not entirely new - it has been discussed for example in [8, 6].) Under our assumptions the use of a modified β-function is therefore simply equivalent to a specific choice of the renormalization scheme.

Let us note that the QCD predictions obtained with the modified β-function would still depend on low order RS parameters, just as would be the case with the predictions in the conventional expansion: in NLO the choice of the scheme may be parametrized by r_1, in NNLO by r_1 and c_2. The difference between the choices of these scheme parameters is formally of higher order in the effective coupling than the order of the approximant, but numerical differences in the predictions may be significant. This ambiguity in the finite order predictions may be resolved by fixing the low order scheme parameters according to the so called Principle of Minimal Sensitivity [8]. For the purpose of this note we use a simpler choice of

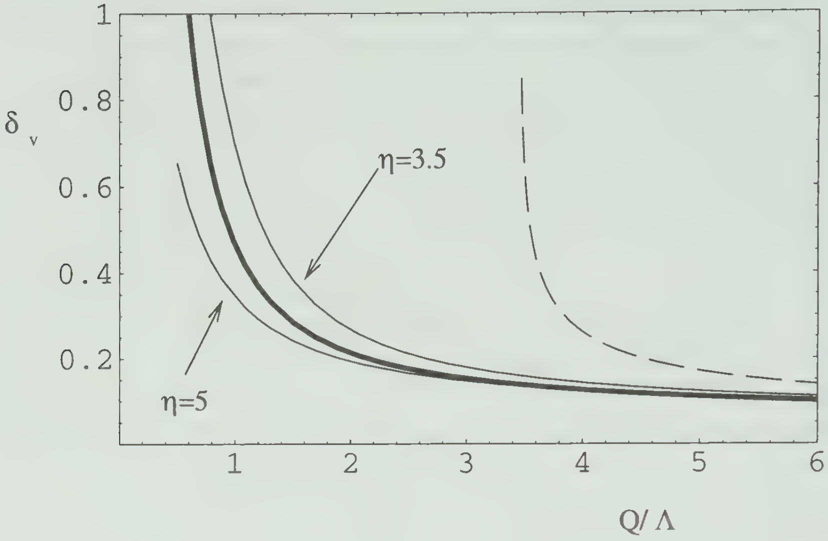

Figure 2. The improved NNLO QCD predictions for δ_V for three "active" quark flavors for $\eta = 3.5$ and $\eta = 5$, compared with the phenomenological expression given in [2] (thick line). The conventional NNLO QCD prediction for δ_V is also shown (dashed line).

$r_1 = 0$ and $r_2 = 0$, which approximates the minimal sensitivity value quite well. Detailed discussion of the optimal choice of the low order RS parameters is beyond the scope of the present note.

The general constraints on the form of the modified β-function mentioned so far are rather weak. In order to further restrict the class of allowed modifications we shall consider the predictions for δ_V, the QCD correction to the potential of interaction between static quarks [9]. The perturbative expansion for this quantity is known up to NNLO. On the other hand, this quantity has been extensively evaluated on the lattice [10], and there exists a useful phenomenological formula [2] (free from singularity at nonzero energy, of course), which seems to be confirmed by the lattice calculations. We shall require that the perturbative NNLO expression for this quantity evaluated using the modified RCC should match the phenomenological expression at low energies.

As a concrete example let us consider the following expression for the modified NNLO β-function:

$$\tilde{\beta}(a) = -\frac{ba^2}{(1+\eta a)^2}\left[1 + \frac{\tilde{c}_1 a}{1+\eta a} + \frac{\tilde{c}_2 a^2}{(1+\eta a)^2}\right], \tag{3}$$

where

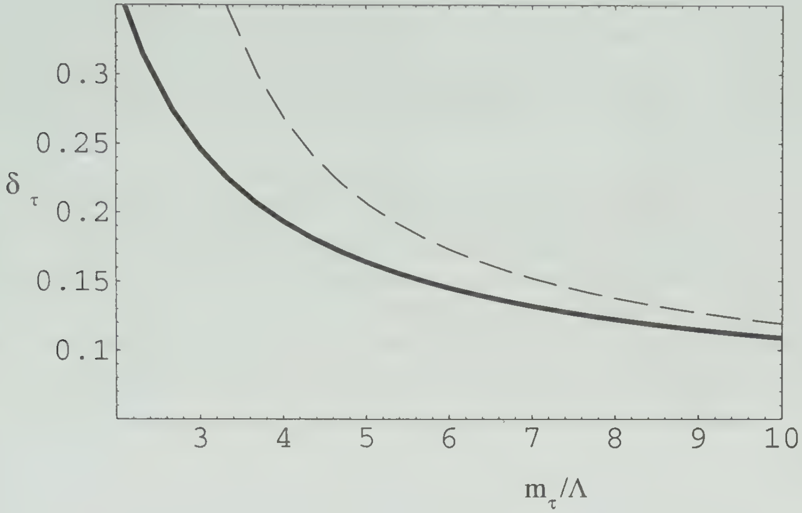

Figure 3. The improved NNLO prediction for the QCD correction to the total hadronic decay rate of the tau lepton (solid line), compared with the conventional NNLO prediction for this quantity (dashed line).

$$\tilde{c}_1 = c_1 + 2\eta, \;\; \tilde{c}_2 = c_2 + 3\eta c_1 + 3\eta^2, \tag{4}$$

and η is an adjustable parameter which controls the deviation of the function from the conventional polynomial form. (It is easy to imagine how a similar modified expression might be defined in arbitrary order.) We show in Figure 1, how the behavior of the modified NNLO β-function compares with the conventional NNLO β-function for two values of c_2 and $\eta = 4$ (we assume that there are three "active" quark flavors).

The numerical value of the improved RCC is determined from the implicit equation resulting from the integration of the RG equation with appropriate asymptotic condition. (All the relevant formulas and conventions may be found for example in [11].) In the case of the modified NNLO β-function (3) the relevant integral may be performed analytically, which greatly simplifies further calculations. The analytic form is too lengthy to be shown here.

In Figure 2 we show, how the NNLO predictions for δ_V depend on the value of η. The requirement that this NNLO prediction should match as closely as possible the phenomenological formula of [2] allows us to make a rough estimate of a sensible value for η; in the following we shall use $\eta = 4$.

We are now in position to obtain improved low energy predictions for other physical quantities of interest. As an example we shall consider improved predictions for QCD correction to the total hadronic tau lepton decay rate [12]. This correction plays important role in the determination of the strong coupling constant. (In fact, the value of the strong coupling constant implied by this quantity was a subject of some controversy [13].) As we may see from Figure 3, the use of a modified RCC results in a substantial reduction of the NNLO QCD correction at a given value of m_τ/Λ.

It is straightforward to apply our modified expansion to perturbative expressions for other physical quantities which are of interest in QCD.

Our discussion of the improved expansion has been very brief and we skipped many details. Full presentation of the method and its applications would be published elsewhere, together with a critical discussion of other approaches. We hope our proposition would be useful in improving the reliability of the QCD predictions at low energies. We also hope our analysis demonstrates, that there are some interesting problems to be solved in perturbation expansion of non-Abelian gauge theories.

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Deformation quantization of Lagrangian fiber bundles

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Abstract. Let (M, ω) be a symplectic manifold. A Lagrangian fiber bundle $\pi : M \to B$ determines a completely integrable system on M. First integrals of this system are the pull-backs of functions on the base of the bundle. We show that for each Lagrangian fiber bundle π there exist star-products on $C^\infty(M)[[\hbar]]$ which do not deform the pointwise multiplication on the subalgebra $\pi^*(C^\infty(B))[[\hbar]]$. The set of equivalence classes of such star-products is in bijection with formal deformations of the symplectic structure ω for which $\pi : M \to B$ remains Lagrangian taken modulo formal symplectomorphisms of M.

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Dedicated to the memory of Moshé Flato

1. Introduction

The phase space of a classical Hamiltonian system is a symplectic manifold (M, ω). Smooth functions on the phase space form the classical algebra of observables. It is a Poisson algebra with, as Lie bracket,

$$\{f, g\} = \omega^{-1}(df \wedge dg), \quad f, g \in C^\infty(M), \tag{1.1}$$

($\omega^{-1} \in \wedge^2 TM$ is the bivector field dual to the two-form ω) and, as commutative product, the pointwise product on $C^\infty(M)$.

A completely integrable classical Hamiltonian system on a $2n$-dimensional symplectic manifold admits n independent integrals which commute with respect to the Poisson bracket. Geometrically this means that away from the singular level surfaces the symplectic manifold M is fibered over an open subset of $\mathbb{R}^n$. More generally for any fiber bundle $\pi : M \to B$, whose fibers are Lagrangian submanifolds of (M, ω) the subalgebra $\pi^*(C^\infty(B))$ of $C^\infty(M)$ is Poisson commutative. This a geometric model for a completely integrable classical system without singularities.

Quantization as a way to pass from classical Hamiltonian mechanics to quantum mechanics can mathematically be regarded as a procedure of replacing the classical algebra of observables (the Poisson algebra of functions on a symplectic manifold) by an associative noncommutative algebra of quantum observables. One approach to quantization is to consider the algebra of quantum observables as a (formal) deformation of the Poisson algebra $(C^\infty(M), \{\cdot, \cdot\})$ of classical observables in the category of associative algebras. For precise definitions see Section 2.

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This approach is known as deformation quantization. It was pioneered by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer [3] who developed the notion of *-product as a formal deformation of the pointwise product on the classical algebra of observables. First examples of *-product can traced back to H. Weyl [18] and J. Moyal [14], who considered the case of symplectic vector spaces (see also J. Vey [16] and F. Berezin [2]).

There were several techniques employed in the study of the deformations of the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$. In [16, 3] a cohomological technique, based on results of [10] was developed. It was successively used in [6] to prove that any symplectic manifold can be quantized. Fedosov [8] found a geometric construction of deformations in the symplectic case. Roughly speaking his approach is based on using Weyl quantization of the cotangent bundle together with a special connection. Then quantized algebra of functions is identified with the space of horizontal sections.

Finally the problem of quantization (classification of formal *-products for a given Poisson structure) was completely solved by Kontsevich [12] for any Poisson manifold. He derived it from the more general result of formality of the differential graded Lie algebra of the Hochschild complex of functions on any smooth manifold.

A Lagrangian fiber bundle $\pi : M \to B$ determines the Poisson commutative subalgebra of classical integrals of motion $\pi^*(C^\infty(B))$ in the algebra of classical observables $C^\infty(M)$. It is natural to ask two questions in this situation:

1. Is it possible to construct a quantization of the Poisson algebra $(C^\infty(M), \{\cdot,\cdot\})$ such that the subalgebra $\pi^*(C^\infty(B))$ will not deform and thus will form an algebra of quantum integrals of motion?

2. If it is possible, then "how many of such quantizations" one can construct?

In this paper, we give answers to both questions using Fedosov's technique. Our results are summarized in the following Theorem.

Theorem 1.1 *(i) The classes of star-products on (M,ω) which are equivalent to products keeping the commutative algebra $\pi^*(C^\infty(B))$ undeformed are in one-to-one correspondence with the deformations of the two-form ω in $H^2(M)[[\hbar]]$ having representatives in $Z^2(M)[[\hbar]]$ with respect to which the bundle $\pi : M \to B$ remains Lagrangian.*

(ii) Any star-product on (M,ω) for which the space $\pi^*(C^\infty(B))[[\hbar]]$ generates a commutative subalgebra of $C^\infty(M)[[\hbar]]$ is equivalent to one that keeps the usual product on $\pi^*(C^\infty(B))[[\hbar]]$ undeformed.

This Theorem provides a positive answer to Question 1. In the framework of Fedosov's quantization the formal two-form associated to a class of star-products on (M,ω) with the described properties is its characteristic form (class). In view of [13], the discussed classes of star-products on (M, ω) are in one-to-one correspondence with formal ($\mathbb{R}[[\hbar]]$) deformations of the symplectic structure ω on M for which $\pi : M \to B$ remains Lagrangian, taken modulo the action of the group of formal symplectomorphisms of (M, ω) starting with the identity diffeomorphism of M.

Remark 1.2 Often a quantization $\star$ of (M, ω) is a priori given (e.g., quantization of a cotangent bundle in terms of differential operators). In this situation one wants to deform (if it is possible) the Poisson commutative algebra $\pi^*(C^\infty(B))$ to a commutative subalgebra of $(CM[[\hbar]], \star)$ by introducing quantum corrections to classical integrals (assuming that the identification between classical and quantum observables is fixed). For this problem, Theorem 1.1 can be used in the following manner. The first step is to check whether the characteristic form of the product $\star$ is of the required type. If it is so, let $*$ be an equivalent product to $\star$ which keeps the algebra $\pi^*(C^\infty(B))$ undeformed and the corresponding formal equivalence be given by the formal power series whose coefficients are differential operators $P = \mathrm{id} + O(\hbar)$. (An iterative procedure for constructing such $*$ is contained in Section 5.) Then $(P^{-1}(\pi^*(C^\infty(B)))[[\hbar]], \star)$ is a commutative subalgebra of $(C^\infty(M)[[\hbar]], \star)$ and the quantum correction to each classical integral of motion $f \in \pi^*(C^\infty(B))$ is $(P^{-1} - \mathrm{id})(f)$.

The paper is organized as follows. In Section 2 we remind some basic definitions and facts about Fedosov's quantization. Section 3 contains the construction of special connections which we use in Sections 4 and 5. The proof of the existence of associative star-products on $C^\infty(M)[[\hbar]]$ preserving given Lagrangian fibration is presented in Section 4. Section 5 is devoted to the classification of such star-products. Theorem 1.1 is a combination of Theorem 4.1 and Theorem 5.1.

One can restrict to the set of star-products on (M, ω) that keep $\pi^*(C^\infty(B))[[\hbar]]$ undeformed and consider a finer equivalence relation given by differential operators that preserve $\pi^*(C^\infty(B))[[\hbar]]$. A classification result in this setting as well as extensions to the case of noncommutative integrable systems will be discussed in a separate publication.

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2. Deformation quantization and Fedosov's construction

2.1. Basic notions

Definition 2.1 A formal local star-product on a Poisson manifold with Poisson bracket $\{\cdot,\cdot\}$ is an associative $\mathbb{R}[[\hbar]]$-linear product on $C^\infty(M)[[\hbar]]$ of the type

$$f*g=fg+\sum_{l=1}^{\infty}\hbar^l Q_l(f,g), \quad f,\, g\in C^\infty(M)[[\hbar]], \tag{2.1}$$

where

1. Q_l are local (i.e., bidifferential) operators,
2. $f*g-g*f=\hbar\{f,g\}+O(\hbar^2), \forall f,g\in C^\infty(M)[[\hbar]]$,
3. $1*f=f*1=f$, i.e., $Q_l(1,f)=Q_l(f,1)=0, \forall f\in C^\infty(M)[[\hbar]], l\geq 1$.

In this paper we will consider only *formal local* star-products and we will call them just star-products.

Definition 2.2 Two star-products $*_1$ and $*_2$ are called equivalent if there exists a local (i.e., differential) $\mathbb{R}[[\hbar]]$-linear operator $P: C^\infty(M)[[\hbar]]\to C^\infty(M)[[\hbar]]$ of the type

$$P=\mathrm{id}+\sum_{l=1}^{\infty}\hbar^l P_l \tag{2.2}$$

for some differential operators P_l on M such that

$$P(f*_1 g)=Pf*_2 Pg, \quad \forall f,g\in C^\infty(M). \tag{2.3}$$

For more details on these definitions we refer to [3, 14].

2.2. The Weyl bundle

Let (M,ω) be a symplectic manifold of dimension $2n$. The Weyl bundle W on M is defined by

$$W=S(T^*M), \tag{2.4}$$

where $S(V)$ denotes the completed symmetric power $S(V)=\coprod_{l=0}^{\infty}S^l(V)$ of a vector space V. For any point $x\in M$ the commutative algebra $S(T_x^*M)$ can be naturally identified with the algebra $\mathcal{J}_0^\infty(T_xM)$ of ∞-jets of real-valued functions on T_xM at 0. More often we will use the bundle

$$W_\hbar=S(T^*M)[[\hbar]]=W[[\hbar]], \tag{2.5}$$

where $\hbar$ is a formal variable and will still call it Weyl bundle of M.

The tangent space T_xM at any $x \in M$ is a symplectic vector space and the Moyal-Weyl product equips any fiber $(W_\hbar)_x$ of $W_\hbar$ with a structure of noncommutative associative algebra. It becomes a graded algebra if we put

$$\deg v = l, \quad \text{for } v \in S^l(T_x^*M), \tag{2.6}$$

$$\deg \hbar = 2. \tag{2.7}$$

The homogeneous component of $(W_\hbar)_x$ in degree l will be denoted by $(W_\hbar^l)_x$. The fiberwise product on $W_\hbar$ determines an associative product on the space $\Gamma(W_\hbar)$ of global sections of $W_\hbar$, denoted by $\circ$.

In local coordinates $(x^1, \ldots, x^{2n}) : U \to \mathbb{R}^{2n}$ the symplectic form ω can be written as

$$\omega = \omega_{jl}\, dx^j \wedge dx^l.$$

Here and later we use the standard convention of summation over repeated indices. Denote by y^j the image of dx^j in $S(T_x^*M)$. Any section $a \in \Gamma(W_\hbar, U)$ is a formal power series:

$$a = \sum_{l \geq 0} \sum_{\alpha} a_{l,\alpha}(x) y^\alpha \hbar^l, \tag{2.8}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{2n})$ is a a multi-index. In this notation the Moyal-Weyl product of two sections a, $b \in \Gamma(W_\hbar, U)$ is

$$a \circ b = \exp\left(\frac{\hbar}{2}\, \omega^{jl}(x)\, \frac{\partial}{\partial y^j} \frac{\partial}{\partial z^l}\right) a(y) b(z)\Big|_{z=y}. \tag{2.9}$$

2.3. Constructing Fedosov's connections

The main idea of Fedosov's quantization is to construct a flat connection in the Weyl bundle $W_\hbar$ for which the exponential map identifies flat sections with smooth functions on M. Then the Moyal-Weyl product (2.9) descends to a star-product on $C^\infty(M)[[\hbar]]$.

Denote by Λ the exterior algebra bundle on M :

$$\Lambda = \wedge(T^*M). \tag{2.10}$$

Moyal-Weyl product on $W_\hbar$ together with the exterior product on Λ equips $W_\hbar \otimes \Lambda$ with a structure of a super algebra. ($W_\hbar$ sits in even degree, Λ^l as usual in even/odd degree depending on the parity of l.) The associative products in $(W_\hbar \otimes \Lambda)_x$, $x \in M$, and $\Gamma(W_\hbar \otimes \Lambda)$ will be again denoted by $\circ$, and the corresponding super bracket by $[\cdot, \cdot]$.

There are two canonical fiberwise endomorphisms of $W_\hbar \otimes \Lambda$, defined locally by

$$\delta(a) = \sum_j dx^j \wedge \frac{\partial a}{\partial y^j} \tag{2.11}$$

and

$$\delta^{-1}(a) = \begin{cases} \frac{1}{p+q}\sum_j y^j \left(\frac{\partial}{\partial x^j} \lrcorner a\right), & \text{for } a \in (W_\hbar^p \otimes \Lambda^q)_x,\ p+q>0, \\ 0, & \text{for } a \in (W_\hbar^0 \otimes \Lambda^0)_x. \end{cases} \tag{2.12}$$

They satisfy

$$\delta^2 = (\delta^{-1})^2 = 0$$

and a Hodge type relation

$$a = (\delta\delta^{-1} + \delta^{-1}\delta)a + a_{00}, \quad \forall a \in (W_\hbar \otimes \Lambda)_x,$$

where a_{00} is the component of a in $(W_\hbar^0 \otimes \Lambda^0)_x$.

Fix a symplectic torsion-free connection on M. Let ∂ denote its covariant derivative. It extends in a natural way to a derivation of $(\Gamma(W_\hbar \otimes \Lambda), \circ)$. In local Darboux coordinates on an open U

$$\partial a = da + \frac{1}{\hbar}[\Gamma, a], \quad a \in \Gamma(W \otimes \Lambda, U), \tag{2.13}$$

where

$$\Gamma = \frac{1}{2}\Gamma_{jkl}y^j y^k dx^l \tag{2.14}$$

and Γ_{jkl} are the Christoffel symbols of the connection in the considered coordinate system.

Consider more general derivations of $(\Gamma(W_\hbar \otimes \Lambda), \circ)$ (or "nonlinear connections" on $W_\hbar$) of the type

$$D = \partial - \delta + \frac{1}{\hbar}\mathrm{ad}(\gamma) \tag{2.15}$$

with

$$\gamma \in \oplus_{p \geq 3}\Gamma(W_\hbar^p \otimes \Lambda^1).$$

One computes

$$D^2 a = \frac{1}{\hbar}[\Omega, a], \quad \forall a \in \Gamma(W_\hbar \otimes \Lambda) \tag{2.16}$$

with

$$\Omega = R + \partial\gamma - \delta\gamma + \frac{1}{\hbar}\gamma^2 + \omega, \tag{2.17}$$

called the Weyl curvature of D (the term ω is for convenience only). Here R is essentially the curvature of ∂

$$R = \frac{1}{4}R_{ijkl}y^i y^j dx^k \wedge dx^l, \quad R_{ijkl} = \omega_{im}R^m_{jkl}. \tag{2.18}$$

The connection D is called a Fedosov's connection when it is flat ($D^2 = 0$). That is when

$$\Omega \in Z(\Gamma(W_\hbar \otimes \Lambda)), \tag{2.19}$$

where $Z(\Gamma(W_\hbar \otimes \Lambda))$ denotes the super center of $(\Gamma(W_\hbar \otimes \Lambda), \circ)$. It is clear that

$$Z(\Gamma(W_\hbar \otimes \Lambda)) = \Omega(M)[[\hbar]], \tag{2.20}$$

where $\Omega(M) = \Gamma(\Lambda, M)$ is the space of differential forms on M. So Equation (2.19) is equivalent to $\Omega \in \Omega^2(M)[[\hbar]]$. In that case Bianchi identities together with Equation (2.16) imply that $D\Omega = d\Omega = 0$, i.e.,

$$\Omega \in Z^2(M)[[\hbar]],$$

where now $Z(M)$ stays for closed forms on M.

Theorem 2.3. (Fedosov) *For any choice of the Weyl curvature* $\Omega \in Z^2(M)[[\hbar]]$, $\Omega = \omega + O(\hbar)$, *there exists a unique Fedosov's connection on* W *with* $\gamma \in \oplus_{p\geq 3}\Gamma(W_\hbar^p \otimes \Lambda^1)[[\hbar]]$ *and* $\delta\gamma = 0$. *The components of the one-form* γ *in the grading* (2.6)–(2.7) *of* $W_\hbar$ *are consecutively determined from*

$$\gamma = \delta^{-1}(-\Omega + \omega + R + \partial\gamma + \frac{1}{\hbar}\gamma^2). \tag{2.21}$$

2.4. Construction of Fedosov's star-products

For a Fedosov's connection D on $W_\hbar$ the space

$$W_D^\hbar = \{a \in \Gamma(W_\hbar) | \, Da = 0\} \tag{2.22}$$

of flat sections of $\Gamma(W_\hbar)$ is a subalgebra of $(\Gamma(W_\hbar), \circ)$.

Given a section $a \in \Gamma(W_\hbar)$ denote by a_0 its component in $S^0(T^*M)[[\hbar]] = C^\infty(M)[[\hbar]]$, i.e., the constant (in y) term in (2.8).

Theorem 2.4. (Fedosov) *For any* $f \in C^\infty(M)[[\hbar]]$ *there exists a unique* $\sigma(f) \in W_D^\hbar$ *such that* $\sigma(f)_0 = f$. *The graded components* $\sigma(f)$ *with respect to the grading* (2.6)–(2.7) *of* $W_\hbar$ *are determined inductively by*

$$\sigma(f) = f + \delta^{-1}\left(\partial(\sigma(f)) + \frac{1}{\hbar}[\gamma, \sigma(f)]\right). \tag{2.23}$$

The above theorem allows to introduce an associative product on $C^\infty(M)[[\hbar]]$ by

$$f * g = (\sigma(f) \circ \sigma(g))_0, \quad f, g \in C^\infty(M)[[\hbar]]. \tag{2.24}$$

One easily sees that it is a deformation of the Poisson structure on (M, ω).

2.5. Classification of star-products

The classification of star-products on a symplectic manifold was obtained in [6] via cohomological methods developed in [3]. In Fedosov's approach it was done in [9, 15, 19]. For comparison between the two approaches see [4]. The classification is summarized in the following theorem.

Theorem 2.5 (*i*) *There exists a bijective correspondence between the set of isomorphism classes of star-products on* (M,ω) *and the classes of formal deformations of the two-form* ω *on* M $(\equiv \{[\Omega] \in H^2(M)[[\hbar]] | \Omega = \omega + O(\hbar)\})$.

(*ii*) *Any star-product is equivalent to a Fedosov's one and the class of this product referred to in part* (*i*) *is the class of the curvature form* $[\Omega]$ *of the corresponding Fedosov's connection (see Theorem 2.3).*

Using Moser's classification of nearby symplectic structures [13], Theorem 2.5 can be rephrased as follows:

The equivalence classes of star-products on (M,ω) *are in one-to-one correspondence with the equivalence classes of formal* $(\mathbb{R}[[\hbar]])$ *deformations of the symplectic structure* ω *on* M.

The equivalence relation in the second set is defined by the action of the group of formal symplectomorphisms of (M,ω) starting with the identity diffeomorphism of M.

2.5.1. *Semi-classical Fedosov's exponential map*
Consider the formal bundle $\mathscr{J}^\infty(M)$ over M whose fiber over $x \in M$ is the set of ∞-jets $\mathscr{J}_x^\infty(M)$ of real-valued functions on M at x. For a symplectic manifold (M,ω) each fiber of $\mathscr{J}^\infty(M)$ has a natural Poisson structure and so does each fiber of the Weyl bundle W. Emmrich and Weinstein [7] found that a semi-classical analog of Fedosov's construction gives a fiberwise isomorphism

$$\exp : \mathscr{J}^\infty(M) \to W. \tag{2.25}$$

Their construction goes as follows. Consider a symplectic torsion-free connection ∂ on M. There exists

$$\gamma \in \oplus_{p\geq 3}\Gamma(W^p \otimes \Lambda^1) \tag{2.26}$$

such that

$$D^0 = \partial - \delta + \{\gamma, .\}_{\text{fib}} \tag{2.27}$$

is a flat connection on W, i.e., $(D^0)^2 = 0$. With $\{\cdot,\cdot\}_{\text{fib}}$ we denote the fiberwise Poisson bracket on W. One such γ can be inductively computed from

$$\gamma = \delta^{-1}\left(R + \partial\gamma + \frac{1}{2}\{\gamma,\gamma\}_{\text{fib}}\right) \tag{2.28}$$

(cf. (2.21)). Here R is the curvature of ∂, as in Equation (2.18). Denote the jet of flat sections of W with respect to D^0 at $x \in M$ by

$$\mathscr{J}_x^\infty(W)_{D^0} = \{a \in \mathscr{J}_x^\infty(W) |\, D^0 a = 0\}. \tag{2.29}$$

For any $f \in \mathscr{J}_x^\infty(M)$ there exists a unique $\bar{f} \in \mathscr{J}_x^\infty(W)_{D^0}$ such that

$$(\bar{f})_0 = f.$$

Iteratively it is computed from

$$\bar{f} = f + \delta^{-1}\left(\partial(\bar{f}) + \{\gamma, \bar{f}\}\right) \tag{2.30}$$

cf. (2.23). At the end one can define

$$\exp_x(f) = \bar{f}(x). \tag{2.31}$$

From Equation (2.30) it follows that the evaluation in the RHS makes sense.

Proposition 2.6 *The map* $\exp : \mathscr{J}_x^\infty(M) \to (W)_x$ *defined by* (2.31) *is a fiberwise Poisson isomorphism.*

Proof. From the fact that D^0 is a differentiation of the Poisson algebra $\Gamma(W)$ and thus differentiation of the algebras $\mathscr{J}_x^\infty(W)$ for any $x \in M$, it follows that exp is a fiberwise Poisson map. The map exp is a fiberwise isomorphism because

$$\exp_x(f) = f + \frac{\partial f}{\partial x^j} y^j + O(y^2) \tag{2.32}$$

in any coordinate system $(x^1, \ldots, x^{2n})$ in a neighborhood of $x \in M$. ■

2.5.2. *A proof of the classification Theorem*

Here we explain some parts of the proof of Theorem 2.5. In particular following P. Xu [19] we show how one obtains from a given star-product $*$ on (M, ω) a Fedosov's star-product equivalent to it. The product $*$ is a local one and thus it induces a star-product on $\mathscr{J}_x^\infty(M)$, $x \in M$. Using the Emmrich-Weinstein exponential map exp we get a fiberwise star-product $\bullet$ on $W_\hbar$:

$$a \bullet b := \exp_x(\exp_x^{-1}(a) * \exp_x^{-1}(b)), \quad \text{for } a,\, b \in (W)_x,\ x \in M. \tag{2.33}$$

This product is a deformation quantization of the fiberwise Poisson algebra structure $\{\cdot,\cdot\}_{\text{fib}}$ on W due to Proposition 2.6. It is fiberwise equivalent to the Moyal-Weyl product $\circ$ (2.9), because TM is a regular Poisson manifold with contractable symplectic leaves. Therefore there exists a differential operator of the form

$$P = \mathrm{id} + \sum_{l=1}^{\infty} \hbar^l P_l,$$

where P_l are differential operators on TM that involve derivatives along the tangent spaces only, such that

$$P(a \bullet b) = P(a) \circ P(b), \quad \forall a, b \in \Gamma(W_\hbar).$$

Let

$$j : C^\infty(M) \to \Gamma(\mathscr{J}^\infty(M))$$

be the natural evaluation map, sending a smooth function to the ∞-jet that it determines at each point of M. Then the composition

$$\rho = P\cdot\exp\cdot j : (C^\infty(M)[[\hbar]],*) \to (\Gamma(W_\hbar),\circ) \tag{2.34}$$

is an injective homomorphism having the following properties:

(i) $(\rho(f))_0 = f$,

(ii) $\rho(f) \equiv f + \delta^{-1}df \ (\mathrm{mod}\,\Gamma(W_\hbar^2))$.

Any Fedosov's connection D gives rise to a map σ_D defined in Theorem 2.4, which satisfies (i)–(ii). P. Xu proved a converse statement.

Theorem 2.7 *Let $*$ be a star-product on M and $\rho : (C^\infty(M)[[\hbar]],*) \to (\Gamma(W_\hbar),\circ)$ be a homomorphism that satisfies (i)–(ii). Then there exists a Fedosov's connection D such that*

$$\rho = \sigma_D.$$

This Theorem applied to the composition (2.34) gives a Fedosov's connection D on $W_\hbar$ and a Fedosov's star-product $*_F$ on M equivalent to the initial one $*$. The class of the curvature form of D is called the characteristic class of the star-product $*$ and is the two-form that appears in Theorem 2.5.

Finally two Fedosov's products are equivalent if and only if their characteristic classes are equal. We will not need a proof of this fact here. The reader can find it in [9, 15].

3. Lagrangian fiber bundles and some associated connections

Let (M,ω) be a symplectic manifold and $\pi : M \to B$ be a Lagrangian fiber bundle, i.e., π is a fiber bundle whose fibers are Lagrangian submanifolds of (M,ω). We assume also that the fibers of π are connected and that for each $f \in C^\infty(B)$ the Hamiltonian vector field $H(\pi^* f)$ is complete. Equivalently the fibers of π should be diffeomorphic to $\mathbb{R}^n/\mathbb{Z}^k$ for some $k \leq n$. We do not require compactness of the fibers thus k can be $< n$.

3.1. Action-angle coordinate charts

In this subsection we review the construction of action-angle coordinate charts on M (see [1, 11, 5]).

For any $b \in B$ define

$$M^b = \{x \in M |\ \pi(x) = b\}. \tag{3.1}$$

T_b^*B acts on M^b in the following way. Let f be a smooth function defined in a neighborhood of b. The action of $(df)_b$ on $x \in M^b$ is defined by

$$\tau((df)_b)x = g_1^{H(\pi^* f)}(x), \tag{3.2}$$

where on the RHS, g denotes the flow on M corresponding to a Hamiltonian vector field (upper index) at a given time (lower index). This flow preserves M^b and thus it depends only on $(df)_b$, so the action is correctly defined. T_b^*B is a commutative group, thus the stabilizers of all points of M^b coincide. Denote the corresponding subgroup of T_b^*B by P_b. It is obvious that $P_b \cong \mathbb{Z}^k$ for some $k \leq n$. Varying b we get a submanifold P of T^*B. It is also clear that it is a Lagrangian submanifold of T^*B. The action of T_b^*B on M^b defines a faithful action of the fibers T_b^*B/P_b of T^*B/P on the fibers M^b of the bundle π.

Let $U \subset B$ be an open set for which there exists a section

$$s : U \to M \tag{3.3}$$

of π $(\pi \circ s = \mathrm{id})$, such that

1. $s(U)$ is a Lagrangian submanifold of M,
2. s is a diffeomorphism onto its image.

The two conditions roughly mean that we consider a Lagrangian submanifold $s(U)$ transversal to the fibration. Any such section determines a symplectic diffeomorphism

$$\pi^{-1}(U) \cong (T^*B/P)|_U, \tag{3.4}$$

where T^*B is equipped with the standard symplectic structure as a cotangent bundle. To prove this simply act with T_b^*B/P_b on $s(b)$ for $b \in U$.

In the case when U is diffeomorphic to a domain in $\mathbb{R}^n$ trivializing $(T^*B/P)|_U$ gives a chart on $\pi^{-1}(U)$ of the type

$$(I^1, I^2, \dots, I^n, \varphi^1, \varphi^2, \dots, \varphi^n) : \pi^{-1}(U) \to O \times (\mathbb{R}^n/\mathbb{Z}^k). \tag{3.5}$$

Here O is an open domain of $\mathbb{R}^n$, which from now on will be denoted by the same letter U. The lattice $\mathbb{Z}^k$ is assumed to be embedded in the standard way in $\mathbb{R}^k \subset \mathbb{R}^n$ spanned by the first k coordinate vectors. The coordinates (I, φ) are called action-angle coordinates, although in the case $k < n$ only $\varphi^1, \dots, \varphi^k$ are angles. Each $b \in B$ has a neighborhood of the above type and thus M can be covered by action-angle charts.

Let

$$(\bar{I}^1, \bar{I}^2, \dots, \bar{I}^n, \bar{\varphi}^1, \bar{\varphi}^2, \dots, \bar{\varphi}^n) : \pi^{-1}(\overline{U}) \to \overline{U} \times (\mathbb{R}^n/\mathbb{Z}^k), \tag{3.6}$$

be another action-angle chart on $\pi^{-1}(\overline{U})$ for some open domain $\overline{U}$ with $U \cap \overline{U} \neq \emptyset$. The change of coordinates on the intersection of the two action-angle charts is of a particularly simple form:

$$\bar{I}^\alpha = f^\alpha(I), \quad \alpha = 1, \dots, n, \tag{3.7}$$

$$\bar{\varphi}^j = A_l^j(I)\varphi^l + B^j(I), \quad i, \alpha = 1, \dots, n, \tag{3.8}$$

where $A_l^j(I)$ is the inverse of the matrix $\frac{\partial \bar{I}^l}{\partial I^j}$ and $f^\alpha, B^1, \dots, B^n$ are some C^∞-functions.

The RHS of (3.8) is linear in φ^l because of the way we have defined the identification (3.4). $B^j(I)$, $j = 1, \dots, n$, are the coordinates of the Lagrangian section $s|_{U \cap \overline{U}}$ of π (see (3.3)) in the action-angle chart $(\bar{I}, \bar{\varphi})$ on $\pi^{-1}(U \cap \overline{U})$:

$$s(\bar{I}) = \left(\bar{I}, B^j\left(I(\bar{I})\right)\right).$$

This imposes some standard differential conditions on the functions $B^j(I)$.

Note also that (3.8) should define a map from $\mathbb{R}^n/\mathbb{Z}^k$ to $\mathbb{R}^n/\mathbb{Z}^k$. This means that $A^j_l(I) + B^j(I)$ should be integers for $j = 1, \dots n$, $l = 1, \dots k$, and thus constants as functions of the action variables I.

3.2. Special symplectic connections

In the second part of this section, we consider certain type of symplectic connections associated to a Lagrangian fiber bundle $\pi : M \to B$. They will be used in Sections 4 and 5 as initial connections ∂ for special Fedosov's connections.

First we give a computational proof of our main theorem and then explain some parts of it in geometric terms.

Theorem 3.1 *There exists a torsion-free symplectic connection* Γ *on* M *such that in each action-angle chart* (I, φ) *as in* (3.5) *the Christoffel symbols of* Γ *satisfy the following properties:*

$$\Gamma^{I^\alpha}_{\varphi^j \varphi^l} = 0, \text{ i.e., } \Gamma_{\varphi^j \varphi^l \varphi^m} = 0, \tag{3.9}$$

$$\Gamma^{\varphi^m}_{\varphi^j \varphi^l} = \Gamma^{I^\beta}_{I^\alpha \varphi^j} = 0, \text{ i.e., } \Gamma_{I^\alpha \varphi^j \varphi^l} = 0, \tag{3.10}$$

$$\Gamma^{\varphi^l}_{I^\alpha \varphi^j}, \Gamma^{I^\gamma}_{I^\alpha I^\beta} \text{ (and thus also } \Gamma_{I^\alpha I^\beta \varphi^j}\text{) do not depend on } \varphi^m, \tag{3.11}$$

$$\Gamma^{\varphi^j}_{I^\alpha I^\beta}, \text{ (and thus also } \Gamma_{I^\alpha I^\beta I^\gamma}\text{) are linear in } \varphi^m. \tag{3.12}$$

$\forall j, k, l, m, \alpha, \beta$, *and* $\gamma = 1, \dots, n$.

Here and after, Greek letters will be used as indices for action coordinates and Latin letters as indices for angle coordinates.

Proof. We construct a connection with the required properties by chart extension. Suppose that we have defined the connection on a union $\pi^{-1}(\mathscr{U})$ of action-angle charts $\pi^{-1}(U)$ as in (3.5) and we would like to extend it to $\pi^{-1}(\overline{U})$, with action-angle coordinates as in (3.6). The Christoffel symbols for Γ on it will be defined consecutively in the order listed in the statement of the theorem.

(1) Because of the transformation rules (3.7)–(3.8) the symbols $\Gamma^{\bar{I}^\alpha}_{\bar{\varphi}^j \bar{\varphi}^l}$ of Γ should satisfy

$$\Gamma^{\bar{I}^\alpha}_{\bar{\varphi}^j \bar{\varphi}^l} = \sum_{j', l', \alpha'} \frac{\partial \bar{I}^\alpha}{\partial I^{\alpha'}} \frac{\partial \varphi^{j'}}{\partial \bar{\varphi}^j} \frac{\partial \varphi^{l'}}{\partial \bar{\varphi}^l} \Gamma^{I^{\alpha'}}_{\varphi^{j'} \varphi^{l'}}$$

on $\pi^{-1}(U \cap \overline{U})$, for $U \subset \mathscr{U}$. But Γ satisfies (3.9) on $\pi^{-1}(U)$, so we are free to put $\Gamma^{\bar{I}^\alpha}_{\bar{\varphi}^j \bar{\varphi}^l} = 0$ on $\pi^{-1}(\overline{U})$.

(2) Similarly, as in part (1), we can define

$$\Gamma^{\bar{\varphi}^m}_{\bar{\varphi}^j\bar{\varphi}^l} = \Gamma^{\bar{I}^\beta}_{\bar{I}^\alpha\bar{\varphi}^j} = 0,$$

on $\pi^{-1}(\overline{U})$ and this will be consistent with the definition of Γ on $\pi^{-1}(\mathscr{U})$.

(3) The transformation property that $\Gamma^{\bar{\varphi}^l}_{\bar{I}^\alpha\bar{\varphi}^j}$ should satisfy is

$$\begin{aligned}\Gamma^{\bar{\varphi}^l}_{\bar{I}^\alpha\bar{\varphi}^j} = {} & \sum_{\alpha'j'l'} \frac{\partial I^{\alpha'}}{\partial \bar{I}^\alpha}\frac{\partial \varphi^{j'}}{\partial \bar{\varphi}^j}\frac{\partial \bar{\varphi}^l}{\partial \varphi^{l'}}\Gamma^{\varphi^{l'}}_{I^{\alpha'}\varphi^{j'}} + \sum_{\alpha'j'\beta'} \frac{\partial I^{\alpha'}}{\partial \bar{I}^\alpha}\frac{\partial \varphi^{j'}}{\partial \bar{\varphi}^j}\frac{\partial \bar{\varphi}^l}{\partial I^{\beta'}}\Gamma^{I^{\beta'}}_{I^{\alpha'}\varphi^{j'}} \\ & + \sum_{\alpha'j'\beta'} \frac{\partial \varphi^{m'}}{\partial \bar{I}^\alpha}\frac{\partial \varphi^{j'}}{\partial \bar{\varphi}^j}\frac{\partial \bar{\varphi}^l}{\partial \varphi^{l'}}\Gamma^{\varphi^{l'}}_{\varphi^{m'}\varphi^{j'}} + \sum_{\beta'j'm'} \frac{\partial \varphi^{m'}}{\partial \bar{I}^\alpha}\frac{\partial \varphi^{j'}}{\partial \bar{\varphi}^j}\frac{\partial \bar{\varphi}^l}{\partial I^{\beta'}}\Gamma^{I^{\beta'}}_{\varphi^{m'}\varphi^{j'}} \qquad (3.13) \\ & + \sum_{l'} \frac{\partial^2 \varphi^{l'}}{\partial \bar{I}^\alpha \partial \bar{\varphi}^j}\frac{\partial \bar{\varphi}^l}{\partial \varphi^{l'}}\end{aligned}$$

on $\pi^{-1}(U \cap \overline{U})$, for $U \subset \mathscr{U}$. Equations (3.9)–(3.11) imply that on $\pi^{-1}(U \cap \overline{U})$ $\Gamma^{I^{\beta'}}_{\varphi^{m'}\varphi^{j'}} = \Gamma^{I^{\beta'}}_{I^{\alpha'}\varphi^{j'}} = \Gamma^{\varphi^{l'}}_{\varphi^{m'}\varphi^{j'}} = 0$ and $\Gamma^{\varphi^{l'}}_{I^{\alpha'}\varphi^{j'}}$ do not depend on φ. From this it is clear that the RHS of (3.13) is a function on $\pi^{-1}(U \cap \overline{U})$, which does not depend on φ (see also (3.7)–(3.8)). It can be used to define $\Gamma^{\bar{\varphi}^l}_{\bar{I}^\alpha\bar{\varphi}^j}$ on $\pi^{-1}(\mathscr{U} \cap \overline{U})$, and at the end we can define them on $\pi^{-1}(\overline{U})$ by any continuation from the closure of $\pi^{-1}(U \cap \overline{U})$.

One deals with the symbols $\Gamma^{\bar{I}^\gamma}_{\bar{I}^\alpha\bar{I}^\beta}$ in the same way.

(4) To define $\Gamma^{\bar{\varphi}^j}_{\bar{I}^\alpha\bar{I}^\beta}$ on $\pi^{-1}(\overline{U})$ so that (3.12) holds, one just repeats the construction from (3).

From this construction we can get a torsion-free symplectic connection satisfying (3.9)–(3.12). This can be done in two ways. First, it is easy to keep on each step the symbols Γ_{xyz} completely symmetric. The second option is to construct a connection Γ with the stated properties for Γ^z_{xy} and then to symmetrize Γ_{xyz}. The result is a connection with all the properties of Theorem 3.1. ■

Remark 3.2 One can interpret Properties (3.9)–(3.11) in a more geometric language.

For each $x \in M, b \in B$, $T_xM^b \subset T_xM$ can be canonically identified with T^*_bB, by linearization of the action of T^*_bB/P_b on M^b. Denote this identification by

$$i_x : T_xM^b \to T^*_bB.$$

Let Γ_B and Γ_M be two connections on B and M respectively with the following property.

If $y \in M_{b'}, b' \in B$ and γ is a curve connecting x and y then

$$\mathscr{T}^M_\gamma(e) = i_y^{-1}\left(\mathscr{T}^B_{\pi(\gamma)}(i_x(e))\right), \qquad (3.14)$$

where $\mathscr{T}^M$ and $\mathscr{T}^B$ denote the parallel translations of tangent vectors to M and cotangent vectors to the base B along the specified curves (corresponding to Γ_M and Γ_B).

Such a connection Γ_M satisfies Properties (3.9)–(3.10) because for $\gamma \subset M^b$

$$\mathscr{T}^B_\gamma \equiv i_y^{-1} i^x : T_x M^b \cong T_y M^b,$$

i.e., M^b is a flat submanifold. Property (3.11) also holds for Γ_M, because the parallel translation $\mathscr{T}^M$ of vectors in $T_x M^b$ comes from the translation $\mathscr{T}^B$ on the base (see (3.14)).

Equation (3.14) can be actually used to define Γ_M on $T_x M^b$, and it will satisfy (3.9)–(3.11). This argument can replace parts (1)–(3) of the above proof.

Remark 3.3 P is a covering of B and we can define parallel translation of the vectors of the lattice $P_b \subset T_b^* B$ along any curve in B. Because of the nature of P (P_b is the kernel of the action τ (3.2) of $T_b^* B$ on M_b) this can be extended by linearity to parallel translation of vectors in span $P_b \subset T_b^* B$. In the special case when π has compact fibers ($k = n$) it gives a torsion-free flat connection on B. Using it as described in the previous remark we get a connection on M for which all Christoffel symbols from (3.9)–(3.11) vanish.

4. Existence of star-products preserving a Lagrangian fiber bundle

The goal of this section is to prove the existence of nontrivial deformations of $C^\infty(M)$, that keep $\pi^*(C^\infty(B))$ undeformed. In the next section we show that up to equivalence of star-products these are all deformations of $C^\infty(M)$ in which $\pi^*(C^\infty(B))$ generates a commutative subalgebra.

Theorem 4.1 *Let $\Omega \in Z^2(M)[[\hbar]]$ be a formal deformation of the symplectic form ω :*

$$\Omega = \omega + O(\hbar), \tag{4.1}$$

such that $\pi : M \to B$ is a Lagrangian fibration with respect to Ω, that is

$$\Omega\Big|_{\pi^{-1}(b)} = 0, \quad \forall b \in B. \tag{4.2}$$

Then there exists a Fedosov's star-product with characteristic class $[\Omega]$ that keeps $\pi^*(C^\infty(B))$ undeformed.

Denote by $\overline{W}_B$ the Weyl bundle on B and by $\Gamma(W_B)$ the pull-back to M of the space of its global sections:

$$\Gamma(W_B) = \pi^*(\Gamma(\overline{W}_B)) \subset \Gamma(W_M). \tag{4.3}$$

Consider also the conormal bundle of the fibration π

$$(N_\pi^*)_x = \{v \in T_x^*M \mid < v, e >= 0, \forall e \in T_xM^b\}, \quad \text{for } x \in M^b,\ b \in B.$$

We will call

$$W_\pi = S(N_\pi^*) \subset W_M \tag{4.4}$$

the Weyl bundle of the fibration π. Also denote $W_{B,\hbar} = W_B[[\hbar]]$, $W_{\pi,\hbar} = W_\pi[[\hbar]]$, and $W_{M,\hbar} = W_M[[\hbar]]$. The following inclusions are obvious

$$\Gamma(W_B) \subset \Gamma(W_\pi) \subset \Gamma(W_M) \tag{4.5}$$

and clearly $\Gamma(W_\pi)$ is an undeformed subalgebra of $\Gamma(W_{M,\hbar})$ with respect to the fiberwise Moyal-Weyl product (2.9).

Locally, in action-angle coordinates (I, φ) on some $\pi^{-1}(U)$ as in Equation (3.5) denote by J^β and ψ^j the images of dI^β and $d\varphi^j$ in $W_M = S(T^*M)$ for $\beta, j = 1, \dots, n$. In this notation $\Gamma(W_{\pi,\hbar}, \pi^{-1}(U))$ consists of sections (2.8) that are power series in J^β only and $\pi^*(\Gamma(\overline{W}_{B,\hbar}, U))$ of the subspace of sections for which, in addition, $a_{k,\alpha}$ depend on the action coordinates I^β only.

In Subsection 4.1 we show that for any closed two-form Ω as in the statement of Theorem 4.1 there exists a closed form Ω' of a special type cohomologous to it. In the second subsection we deduce Theorem 4.1 from a stronger result stating that Fedosov's construction applied to Ω' exponentiates $\pi^*(C^\infty(B))$ to a subalgebra of $\Gamma(W_{B,\hbar})$. The latter, as we have remarked above, is an undeformed subalgebra of $(\Gamma(W_{M,\hbar}), \circ)$.

4.1. Construction of special closed two-forms

Denote by $\Gamma(W_{M,\hbar} \otimes \Lambda)^{\mathrm{pol}}$ the subalgebra of $\Gamma(W_{M,\hbar} \otimes \Lambda)$ consisting of those sections which in any action-angle chart (3.5) are formal power series

$$\sum_{l=0}^{\infty} \sum_{\alpha,\gamma} \hbar^l f_{a,\gamma,l}(I, \varphi) y^\alpha dx^\gamma \tag{4.6}$$

with $f_{a,\gamma,l}(I, \varphi)$ being polynomials in the variables $\varphi^{k+1}, \dots, \varphi^n$, where $x = (I, \varphi)$, $y = (J, \psi)$, and $dx^\gamma = dx^{\gamma_1} \wedge \dots \wedge dx^{\gamma_s}$ (see also the remarks related to Equation (3.5)). The above requirement does not depend on the choice of a particular action-angle coordinate system (I, φ) on an open $\pi^{-1}(U)$ because of the transformation rules (3.7)–(3.8). The same formulas make possible to define a filtration on $\Gamma(W_{M,\hbar} \otimes \Lambda)^{\mathrm{pol}}$ by letting (locally)

$$\deg \varphi^j = \deg d\varphi^j = \deg \psi^j = 1, \quad j = 1, \dots, n. \tag{4.7}$$

Denote the part of $\Gamma(W_{M,\hbar} \otimes \Lambda)^{\mathrm{pol}}$ of deg $\leq l$ by

$$\Gamma(W_{M,\hbar} \otimes \Lambda)^{(l)} \tag{4.8}$$

(not to be mistaken with the grading (2.6)). In the same way we define the subspaces $\Gamma(W_{M,\hbar})^{\mathrm{pol}}$ and $\Gamma(\Lambda)^{\mathrm{pol}}$ of $\Gamma(W_{M,\hbar}\otimes\Lambda)^{\mathrm{pol}}$ and restrict the filtration (4.8) to them.

We are ready to state the main result of this subsection.

Proposition 4.2 *Let $\Omega\in Z^2(M)$ such that*

$$\Omega\Big|_{\pi^{-1}(b)}=0,$$

for all $b\in B$. Then there exists $\Omega'\in Z^2(M)$ cohomologous to Ω such that

$$\Omega'\in\Gamma(\Lambda^2)^{(1)}. \tag{4.9}$$

(Then also automatically $\Omega'|_{\pi^{-1}(b)}=0$, $\forall b\in B$.)

Both the above result and the Lemma that we need for its proof are well known but we will sketch two elementary proofs of them for completeness.

Lemma 4.3 *(i) Let $\alpha\in Z^2(U\times\mathbb{R}^n)$ for an open domain U in $\mathbb{R}^n$. Assume that*

$$\alpha\Big|_{\pi_1^{-1}(u)}=0,\quad \forall u\in U, \tag{4.10}$$

where $\pi_1 : U\times\mathbb{R}^n\to U$ is the projection on the first factor. Then there exists a one-form β such that

$$\beta\Big|_{\pi_1^{-1}(u)}=0,\quad \forall u\in U\quad \text{and}\quad d\beta=\alpha. \tag{4.11}$$

(ii) If $\beta\in Z^1(U\times\mathbb{R}^n)$ is such that

$$\beta\Big|_{\pi_1^{-1}(u)}=0,\ \forall u\in U,$$

for an open subset $U\subset\mathbb{R}^n$, then

$$\beta=\pi^*(\beta_1),$$

for some $\beta_1\in Z^1(U)$.

Proof. Part (i): Denote by $(I^1,\dots,I^n)$ and $(\varphi^1,\dots,\varphi^n)$ the coordinates on U and $\mathbb{R}^n$ respectively. If

$$\alpha=f_{jl}(I,\varphi)dI^j\wedge dI^l+g_{jl}(I,\varphi)d\varphi^j\wedge dI^l$$

then $\alpha^0=f_{jl}(I,0)dI^j\wedge dI^l$ is closed in U and there exists a one-form $\beta^0=a_l(I)dI^l$ such that $d\beta^0=\alpha^0$. Using the assumption that α is closed it is easy to prove that

$$\beta=\beta^0+\left(\int_0^{\varphi}g_{jl}(I,\phi)d\phi^j\right)dI^l$$

satisfies (4.11).

Part (ii): If $\beta = b_l(I,\varphi)dI^l$ then $d\beta = 0$ implies $d_\varphi b_l(I,\varphi) = 0$, $\forall I \in U$. Thus $b_l(I,\varphi)$ depends on I only and we can put $\beta_1 = b_l(I)dI^l$. ■

Proof of Proposition 4.2. We construct a two-form Ω', satisfying (4.9) and a one-form γ such that $\Omega - \Omega' = d\gamma$ by chart extension.

Let (I,φ) be an action-angle chart on some $\pi^{-1}(U) \cong U \times (\mathbb{R}^n/\mathbb{Z}^k)$. Denote

$$\Omega_U = \Omega\Big|_{\pi^{-1}(U)}$$

and let its pull-back to $U \times \mathbb{R}^n$ be

$$\widetilde{\Omega}_U = f_{jl}(I,\varphi)dI^j \wedge dI^l + g_{jl}(I,\varphi)d\varphi^j \wedge dI^l \tag{4.12}$$

(here f_{jl} and g_{jl} are $\mathbb{Z}^k$ periodic in $\varphi^1,\dots,\varphi^k$). Lemma 4.3 applied to $\widetilde{\Omega}_U$ gives a one-form $\tilde{\beta}_U = a_l(I,\varphi)dI^l$ such that $d\tilde{\beta}_U = \widetilde{\Omega}_U$. For any l and $I = c = (c^1,\dots,c^n) =$ const.,

$$g_{jl}(c,\varphi)d\varphi^j \in Z^2(\mathbb{R}^n/\mathbb{Z}^k)$$

because Ω is closed. $H^2(\mathbb{R}^n/\mathbb{Z}^k) \cong \mathbb{R}^k$ and there exists a one-form with coefficients that do not depend on the angle variables φ

$$\sum_j a'_{jl}(c)d\varphi^j \sim g_{jl}(c,\varphi)d\varphi^j, \tag{4.13}$$

where the first sum is over $1 \le j \le k$, i.e., just coordinates $\varphi^1,\dots,\varphi^k$ related to the lattice $\mathbb{Z}^k$ appear. Then

$$\gamma_U := (a_l(I,\varphi) - \sum_j a'_{jl}(I)\varphi^j)dI^l \tag{4.14}$$

descends to a one-form on $U \times \mathbb{R}^n/\mathbb{Z}^k$, and so does

$$\Omega'_U := \Omega_U - d\gamma_U = d((\sum_j a'_{jl}(I)\varphi^j)dI^l) \tag{4.15}$$

(to a two-form on $U \times \mathbb{R}^n/\mathbb{Z}^k$). Clearly

$$\Omega'_U \in \Gamma(\Lambda^2, \pi^{-1}(U))^{(1)}.$$

Suppose now that we have already constructed Ω' and γ on a union of action-angle charts and we want to extend them to an open $\pi^{-1}(U)$ on M as in (3.5). The intersection of $\pi^{-1}(U)$ with this union is $\pi^{-1}(U^0)$, for some open $U^0 \subset U$. Denote the restriction of γ to $\pi^{-1}(U^0) \cong U^0 \times \mathbb{R}^n/\mathbb{Z}^k$ by γ_{U^0}. The second part of Lemma 4.3 together with the transformation rules (3.7)–(3.8) imply that

$$\gamma_{U^0} = (a_l(I,\varphi) - a'_{jl}(I)\varphi^j - d_l(I))dI^l \tag{4.16}$$

for some functions $d_l(I)$ on U^0 (the sum over j is from 1 to k again). Finally we can extend $d_l(I)$ to C^∞-functions on U from the closure of U^0 and define an extension γ_U of γ_{U^0} to $\pi^{-1}(U)$ by the same formula (4.16). Ω'_{U^0} also can be extended to $\pi^{-1}(U)$ by $\Omega'_U = \Omega_U - d\gamma_U$. Locally on $\pi^{-1}(U)$

$$\Omega'|_U = d(a_{jl}(I)\varphi^j + d_l(I)) \wedge dI^l \in \Gamma(\Lambda^2, \pi^{-1}(U))^{(1)}.$$

■

4.2. Construction of star-products preserving a Lagrangian fiber bundle

The next theorem is the main result in this subsection. It makes Theorem 4.1 more precise, the latter is a consequence of it due to Proposition 4.2.

Theorem 4.4 *Let $\Omega' \in Z^2(M)[[\hbar]]$ be a formal deformation of ω, $(\Omega' = \omega + O(\hbar))$ with the property from Proposition 4.2*

$$\Omega' \in \Gamma(\Lambda^2 M[[\hbar]])^{(1)}. \tag{4.17}$$

Then any Fedosov's connection D with curvature Ω' based on a symplectic connection ∂ with the properties of Theorem 3.1 exponentiates $\pi^*(C^\infty(B))[[\hbar]]$ to a subalgebra of $\Gamma(W_{B,\hbar})$:

$$\sigma_D(\pi^*(C^\infty(B))[[\hbar]]) \subset \Gamma(W_{B,\hbar}).$$

We will need a simple Lemma.

Lemma 4.5 *Assume that $a \in \Gamma(W_{M,\hbar})^{(q)}$ and $b \in \Gamma(W_{M,\hbar})^{(r)}$, then*

(i) $[a,b] \in \Gamma(W_{M,\hbar})^{(q+r-1)}$,

(ii) $\partial a \in \Gamma(W_{M,\hbar})^{(q)}$, for any connection ∂ with the properties of Theorem 3.1,

(iii) δa and $\delta^{-1} a \in \Gamma(W_{M,\hbar})^{(q)}$.

Proof. Parts (i) and (iii) are trivial. Part (ii) follows from

$$\begin{aligned}
\partial I^\alpha &= \Gamma^{I^\alpha}_{I^\beta I^\gamma} J^\beta dI^\gamma \in \Gamma(W_{M,\hbar})^{(0)},\\
\partial \varphi^l &= \Gamma^{\varphi^l}_{I^\beta \varphi^j} J^\beta d\varphi^l + \Gamma^{\varphi^l}_{\varphi^j I^\beta} \psi^j dI^\beta + \Gamma^{\varphi^l}_{I^\alpha I^\beta} J^\alpha dI^\beta \in \Gamma(W_{M,\hbar})^{(1)},
\end{aligned}$$

for any connection ∂ satisfying (3.9)–(3.12). ■

Proof of Theorem 4.4. Equation (2.13) implies that the curvature form R of ∂ (see (2.18)) in local Darboux coordinates is

$$R = \partial\Gamma + \frac{1}{\hbar}(\Gamma \circ \Gamma),$$

where $\Gamma = (1/2)\Gamma_{ijl}y^iy^kdx^l$ is the connection form of ∂ in the considered coordinate system. Clearly from (3.9)–(3.12) we have

$$\Gamma \in \Gamma((W \otimes \Lambda^1)^{\mathrm{pol}}, \pi^{-1}(U))^{(1)}$$

and from Lemma 4.5 we get that

$$R \in \Gamma(W_M \otimes \Lambda^2)^{(1)}$$

(see Lemma 4.5). Using the same Lemma again from Equation (2.21) we obtain

$$\gamma \in \Gamma(W_{M,\hbar} \otimes \Lambda^1)^{(1)}. \tag{4.18}$$

Finally Lemma 4.5 applied to (2.23) gives

$$\sigma(f) \in \Gamma(W_{M,\hbar})^{(0)} = \Gamma(W_{B,\hbar}), \quad \forall f \in \pi^*(C^\infty(B)).$$

■

4.3. Another proof of the existence result

Here we sketch another proof of Theorem 4.1 that does not use Proposition 4.2. The idea is to use Fedosov's construction with $\Omega = \omega + O(\hbar) \in H^2(M)[[\hbar]]$ instead of ω.

Consider M as a $C^\infty[[\hbar]]$-manifold. That is we take $\mathscr{O}_M^\hbar = \mathscr{O}_M[[\hbar]]$ as structure sheaf on M ($\mathscr{O}_M$ is the sheaf of C^∞-functions on M). For any open $U \subset M$ $\Gamma(\mathscr{O}_M^h, U) = \Gamma(C^\infty(M), U)[[\hbar]]$ and a coordinate system on U is specified by the images of the coordinate functions $x_1, \dots, x_{2n}$ under an isomorphism

$$F : C^\infty(U_o)[[\hbar]] \to \Gamma(\mathscr{O}_M^h, U),$$

for an open $U_o \subset \mathbb{R}^{2n}$. Any composition $f(g_0 + \hbar g_1 + \dots)$ for some C^∞-functions $f, g_0, g_1, \dots$ is computed using the Taylor rule

$$f(g_0 + \hbar g_1 + \cdots) = \sum_{l=0}^{\infty} f^{(l)}(g_0)G^l/l!, \quad G = \hbar g_1 + \hbar^2 g_2 + \cdots.$$

One can consider (M, Ω) as a symplectic $C^\infty[[\hbar]]$ manifold and prove the existence of a covering of it with action-angle charts. Let $\pi : M \to B$ be a fiber bundle which is Lagrangian with respect to Ω in the usual sense. Using Lemma 4.3 one constructs a symplectic connection $\bar{\Gamma}$ on (M, Ω) with covariant derivative $\bar{\partial} : \Gamma(TM) \to \Gamma(TM)[[\hbar]]$, that satisfies properties (3.9)–(3.12) for the data M, π, Ω. It is obvious that the constant term of $\bar{\partial}$

$$\partial : TM \to TM$$

is a connection with the properties of Theorem 3.1 for the data M, π, ω.

Now we can apply Fedosov's construction with ω and ∂ replaced by Ω and $\bar{\partial}$. $W_{M,\hbar}$ will be equipped with the Moyal-Weyl product corresponding to Ω, let us denote it by $\bar{\circ}$. Consider a Fedosov's connection for $(W_{M,\hbar}, \bar{\circ})$

$$\overline{D} = \bar{\partial} - \delta + \frac{1}{\hbar}\mathrm{ad}(\bar{\gamma}),$$

with curvature form (same) Ω. As in the proof of Theorem 4.4 (see also Lemma 4.5) one proves that it induces a star-product $*$ on M which keeps $\pi^*(C^\infty(B))$ undeformed. In the remaining part of this subsection we show that $*$ has characteristic class $[\Omega]$.

One can find a fiberwise linear map $F : TM \to TM[[\hbar]]$, such that $F(\omega) = \Omega$. Then

$$F^* : (W_{M,\hbar}, \bar{\circ}) \to (W_{M,\hbar}, \circ)$$

is an isomorphism. Extend F^* to $W_{M,\hbar} \otimes \Lambda$ trivially on the second factor. Clearly

$$F^*\bar{\partial}(F^*)^{-1} = \partial + \mathrm{ad}_\circ(\alpha) \text{ and}$$
$$F^*\delta(F^*)^{-1} = \delta + \mathrm{ad}_\circ(\beta),$$

for some one-forms α and β with values in $S^2(T^*M)$ and $S^1(T^*M)$ respectively. Here $\mathrm{ad}_\circ$ refers to the the Lie bracket associated with the product $\circ$. As a consequence of this we get that

$$\begin{aligned} D &:= (F^*)^{-1}\overline{D}F^* \\ &= \partial - \delta + \frac{1}{\hbar}\mathrm{ad}_\circ(F^*(\bar{\gamma}) + \hbar\alpha - \hbar\beta), \end{aligned}$$

is a Fedosov's connection for $(W_{M,\hbar}, \circ)$. By a straightforward computation one shows that the Weyl curvature of D is Ω and F^* induces isomorphism between $W_{\overline{D}}$ and W_D. It is now clear that the star-products coming from $\overline{D}$ and D are equivalent and have characteristic class $[\Omega]$.

5. Classification of star-products preserving a Lagrangian fiber bundle

The results of this Section are summarized in the following theorem.

Theorem 5.1 *Let $*$ be a star-product on the symplectic manifold (M, ω) for which $\pi^*(C^\infty(B))$ generates a commutative subalgebra of $C^\infty(M)[[\hbar]]$. Then its characteristic class has a representative $\Omega \in Z^2(M)[[\hbar]]$ such that $\pi : M \to B$ is Lagrangian with respect to Ω :*

$$\Omega\Big|_{\pi^{-1}(b)} = 0, \quad \forall b \in B. \tag{5.1}$$

In addition, there is always a Fedosov's star-product on (M, ω) that is equivalent to $*$ and keeps $\pi^*(C^\infty(B))$ undeformed.

The first step towards our proof of Theorem 5.1 is a version of a well-known fact, which states that any commutative deformation of the algebra of functions on a smooth manifold is equivalent to a trivial one.

Lemma 5.2 *Let* $*$ *be a star-product on* (M,ω) *for which the algebra generated by* $\pi^*(C^\infty(B))$ *is commutative. Then there exists an equivalent product* $\star$ *on* (M,ω) *that keeps* $\pi^*(C^\infty(B))$ *undeformed.*

We will skip the proof of this lemma as it is similar to the proof of Proposition 5.4 below.

The second step in our proof of Theorem 5.1 is to exponentiate $\star$ to a fiberwise star-product $\bullet$ on $W_\hbar$ using Emmrich-Weinstein construction [7] reviewed in Subsection 2.5.1 with a symplectic connection ∂ having the properties of Theorem 3.1.

Lemma 5.3 *Let* $\star$ *be a star-product on* (M,ω) *that keeps* $\pi^*(C^\infty(B))$ *undeformed. Denote the fiberwise star-product* $\bullet$ *on* $W_{M,\hbar}$ *obtained from* $\star$ *by the exponentiation construction of Subsection 2.5.1 using a symplectic connection* ∂ *with the properties of Theorem 3.1 (see Equation* (2.33)*). Then* $\bullet$ *keeps each fiber of* $W_{\pi,\hbar}$ *undeformed.*

Proof. In the exponentiation procedure from Subsection 2.5.1 one first computes the one-form γ for the nonlinear flat connection D^0 on W (see (2.27)). It is done inductively using (2.28). Similarly, as in the proof of Theorem 4.4 (see especially (4.18)), one gets

$$\gamma \in \Gamma(W_M)^{(1)}.$$

Thus

$$\exp(f) \in \Gamma(W_M)^{(0)} = \Gamma(W_B), \quad \forall f \in \pi^*(C^\infty(B)).$$

(see Equation (2.31) for the definition of exp). The closure of

$$\operatorname{span}\{\exp(f)_x | f \in \pi^*(C^\infty(B))\} \tag{5.2}$$

in $S(T^*_x M)$ is $S((N^*_\pi)_x) = (W_\pi)_x$, $\forall x \in M$, which proves the Lemma. ∎

The next step is to study the equivalence between the product $\bullet$ and the Moyal-Weyl product $\circ$ (2.9).

Proposition 5.4 *There exists a leafwise differential operator* P *on* TM *of the type*

$$P = \mathrm{id} + \hbar P_1 + \hbar^2 P_2 + \cdots, \tag{5.3}$$

for some differential operators P_j *on* TM *with coefficients in* W_M *such that*

$$P(a \bullet b) = P(a) \circ P(b), \quad \forall a, b \in (W)_x,\ x \in M, \tag{5.4}$$

and

$$Pa = a, \quad \forall a \in (W_\pi)_x \subset (W)_x. \tag{5.5}$$

Proof. Denote by C^l the bundle over M whose fiber at $x \in M$ consists of (local) Hochschild l-cochains for $(W)_x = \mathcal{J}_0^\infty(T_xM)$ (equipped with the standard fiberwise commutative product). Any such cochain is represented by an l-differential operator on the tangent space T_xM.

Let us define

$$\begin{aligned}(C^l)^s = \{B \in C^l | B &\text{ is represented by an} l - \text{differential operator}\\ &\text{of degree } > 1 \text{ in at least one component or it is}\\ &\text{of degree 1 in any component and } A(B) = 0\},\end{aligned} \tag{5.6}$$

where A denotes the complete antisymmetrization and

$$(C^l)^a = \{B \in C^l | B \text{ is antisymmetric of degree 1 in each component}\}. \tag{5.7}$$

Let $(C^l)^a_\pi$, $(C^l)^s_\pi$, and C^l_π be the subbundles of $(C^l)^a$, $(C^l)^s$, and C^l consisting of those cochains that vanish if one of their arguments is in W_π. It is clear that

$$C^l_\pi = (C^l)^a_\pi \oplus (C^l)^s_\pi. \tag{5.8}$$

By δ and $[\cdot,\cdot]$ (in this proof only) we will denote the (fiberwise) differential and the (super) Lie bracket in the Hochschild complex C. $\alpha \in \Gamma(C^2)$ will denote the cochain corresponding to the fiberwise Poisson bracket $\{\cdot,\cdot\}_{\text{fib}}$ on W_M. The following statements are well known:

(i) $\delta((C^l)^s) \subset (C^{l+1})^s$ and

$$\cdots \xrightarrow{\delta} (C^{l-1})^s \xrightarrow{\delta} (C^l)^s \xrightarrow{\delta} (C^{l+1})^s \xrightarrow{\delta} \cdots$$

is acyclic. In particular

$$(C^1)^s \xrightarrow{\delta} (C^2)^s$$

is an inclusion onto $\operatorname{Ker} \delta_2((C^2)^s \to (C^3)^s)$.

(ii) $(TM, \{\cdot,\cdot\})$ is a regular Poisson manifold with symplectic leaves T_xM, $x \in M$. The restriction of α to each leaf canonically identifies the fiber of $(C^l)^s$ at x with the space of forms $\Omega(T_xM)$ on T_xM with coefficients in $(W)_x = \mathcal{J}_0^\infty(T_xM)$. Under this identification $\{\alpha,.\}$ goes to the fiberwise differential d_{TM} (see [15]).

We prove that there exists a differential operator P as in (5.3) satisfying (5.4) such that

$$P_n \in \Gamma((C^2)_\pi). \tag{5.9}$$

The last equation obviously implies that P also satisfies (5.5). According to (5.8) P_n should decompose as

$$P_n = P_n^s + P_n^a, \quad \text{where } P_n^s \in \Gamma((C^1)^s_\pi) \text{ and } P_n^a \in \Gamma((C^1)^a_\pi).$$

The operator P is constructed iteratively. At the n-th step assuming that (5.4) holds modulo $\hbar^{n-1}$, we define P_n^a and P_{n-1}^s in such a way that (5.4) is true modulo $\hbar^n$.

The equations for P_n^s and P_{n-1}^a are

$$\delta P_n^s = Q_n^s, \tag{5.10}$$

$$[\alpha, P_{n-1}^a] = Q_{n-1}^a. \tag{5.11}$$

with some $Q_n^s \in \Gamma((C^2)_\pi^s)$ and $Q_{n-1}^a \in \Gamma((C^2)_\pi^a)$, which are known from the previous steps and satisfy

$$\delta Q_n^a = 0, \tag{5.12}$$

$$[\alpha, Q_{n-1}^s] = 0. \tag{5.13}$$

According to (i), (5.10) has a unique solution P_n^s in $\Gamma((C^1)^s)$ and this solutions is in $\Gamma((C^1)_\pi^s)$. The existence of a solution $P_{n-1}^a \in \Gamma((C^1)_\pi^a)$ of (5.11) can be proved in the following way. The identification from (ii) associates with Q_{n-1}^a a family of two-forms β_x^2 on $T_xM, x \in M$, with coefficients in $\mathscr{J}_0^\infty(T_xM)$. The restriction of β_x^2 to T_xM^b vanishes for any $x \in M$ (here M^b denotes the fiber of π passing trough x). A formal analog of Lemma 4.3 implies that there exists a family of one forms β_x^1 on TM whose restrictions to T_xM^b vanish and on each tangent space T_xM

$$d\beta_x^1 = \beta_x^2.$$

Applying the identification from (ii) back to β_x^1 plus some standard gluing arguments give the existence of a solution to (5.11) of the desired form. ■

Finally Theorem 5.1 follows directly from the next Theorem, which makes it more precise.

Theorem 5.5 *Let $*$ be a star-product on (M,ω) for which $\pi^*(C^\infty(B))$ generates a commutative subalgebra in $C^\infty(M)[[\hbar]]$. Then there exists an injective homomorphism*

$$\rho : (C^\infty(M)[[\hbar]], *) \hookrightarrow (\Gamma(W_{M,\hbar}), \circ) \tag{5.14}$$

such that

$$\rho(\pi^*(C^\infty(B))) \subset \Gamma(W_{B,\hbar}). \tag{5.15}$$

There exists a Fedosov's connection D on $W_\hbar$ for which

$$\rho = \sigma_D. \tag{5.16}$$

Its Weyl curvature satisfies (5.1). *The corresponding Fedosov's product is equivalent to $*$ and keeps $\pi^*(C^\infty(B))$ undeformed.*

Proof. Most of the work needed is already done. The homomorphism ρ is just the composition of the maps from Lemmas 5.2, 5.3, and Proposition 5.4. It enjoys Properties (i) and (ii) of Subsection 2.5.1 and from Theorem 2.7 we get that there exists a Fedosov connection $D = \partial - \delta + (1/\hbar)\mathrm{ad}(\gamma)$ satisfying (5.16). Then for any $f \in C^\infty(M)$

$$\left(\partial - \delta - \frac{1}{\hbar}\mathrm{ad}(\gamma)\right)\rho(f) = 0, \quad \forall f \in C^\infty(M).$$

We have the freedom to choose the initial connection ∂. If it has the properties of Theorem 3.1, then

$$(\partial - \delta)\rho(f) \in \Gamma(W_{M,\hbar} \otimes \Lambda^1)^{(0)}, \quad \forall f \in \pi^*(C^\infty(B)),$$

and thus

$$[\gamma, \rho(f)] \in \Gamma(W_{M,\hbar} \otimes \Lambda^1)^{(0)}, \quad \forall f \in \pi^*(C^\infty(B)).$$

The operator P from Proposition 5.4 acts trivially on $\pi^*(C^\infty(B))$, so

$$\rho(f) = \exp(f) \in \Gamma(W_{\pi,\hbar}), \quad \forall f \in \pi^*(C^\infty(B)),$$

where exp is the exponential map from the proof of Lemma 5.3. This proves the fact about Fedosov's product in Theorem 5.5. It also implies that the completion of

$$\mathrm{span}\{\rho(f)_x | f \in \pi^*(C^\infty(B))[[\hbar]]\}$$

in $(W_\hbar)_x$ is $(W_{\pi,\hbar})_x$ $(\forall x \in M)$ and therefore that for any open domain $U \subset B$

$$\gamma\big|_U \in \Gamma(W_{M,\hbar} \otimes \Lambda^1, \pi^{-1}(U))^{(1)} + \Gamma(W_{\pi,\hbar} \otimes \Lambda^1, \pi^{-1}(U)). \tag{5.17}$$

It is clear that in local action-angle coordinates (I, φ)

$$(W_\pi)_m = \mathbb{R}[[J^1, \dots, J^n]]$$

and that it is the commutant of itself in

$$\left((W_{M,\hbar})_m = \mathbb{R}[[J^1, \dots, J^n, \psi^1, \dots, \psi^n]], \circ\right).$$

Recall that by J^l and ψ^l we denote the respective images of dI^l and $d\varphi^l$ in $(W)_x$, $x \in M$. At the end the Weyl curvature of D is

$$\begin{aligned} \Omega &= R + \partial\gamma - \delta\gamma + \frac{1}{\hbar}\gamma^2 \\ &= (\gamma \circ \gamma)_0 \end{aligned} \tag{5.18}$$

(see the discussion before Theorem 2.4 for the definition of constant term $(\cdot)_0$). Locally write γ as a sum of two terms according to (5.17). Then nonzero contribution to $(\gamma \circ \gamma)_0$ give terms of the following type only:

$$\gamma_1' \circ \gamma_1'', \text{ for } \gamma_1', \gamma_1'' \in \Gamma(W^j_{M,\hbar} \otimes \Lambda^1)^{(1)}, \quad j = 1 \text{ or } 2, \tag{5.19}$$

$$\gamma_1 \circ \gamma_2, \text{ for } \gamma_1 \in \Gamma(W^1_{M,\hbar} \otimes \Lambda^1)^{(1)}, \quad \gamma_2 \in \Gamma(W_{\pi,\hbar} \otimes \Lambda^1). \tag{5.20}$$

In the case (5.19)

$$\gamma_1' \circ \gamma_1'' \in \Gamma(\Lambda^2)^{(2)}[[\hbar]]$$

and all forms in $\Gamma(\Lambda^2)^{(2)}[[\hbar]]$ satisfy (5.1).

Consider the case (5.20). Locally γ_1 is a polynomial in ψ^j of degree at most 1. The product (5.20) eats out the ψ's and what remains is in $\Gamma(\Lambda, \pi^{-1}(U))^{(0)}[[\hbar]]$. As a one-form it vanishes on M^b, $\forall b \in B$ and so does $\gamma_1 \circ \gamma_2$. This ends the proof of Theorem 5.5. ■

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Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization

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Abstract. For arbitrary compact quantizable Kähler manifolds it is shown how a natural formal deformation quantization (star-product) can be obtained via Berezin-Toeplitz operators. Results on their semi-classical behaviour (their asymptotic expansion) due to Bordemann, Meinrenken, and Schlichenmaier are used in an essential manner. It is shown that the star-product is null on constants and fulfills parity. A trace is constructed and the relation to deformation quantization by geometric quantization is given.

Keywords: quantization, Kähler manifolds, star-products, semi-classical limit, Toeplitz structure

Mathematics Subject Classifications (2000): 58F06, 58F05, 53C55, 32C17, 81S10

Dedicated to the memory of Moshé Flato

1. Introduction

By Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in 1977 the important concept of quantization given by deforming the algebra of functions in "direction" of the Poisson bracket was introduced [1]. Clearly the intuitive concept of $\hbar$-depending "deformation" of classical mechanics into quantum mechanics was around earlier (e.g. Weyl quantization). But in their work a mathematically very precise meaning was given to it.

Since this time the existence of a deformation quantization for every symplectic manifold was established in different ways. Some of the persons involved were De Wilde and Lecomte [11], Fedosov [15], and Omori, Maeda, and Yoshioka [27]. Quite recently this was extended to every Poisson manifold by Kontsevich [21]. Classification results are also available [4, 13, 26, 14, 36].

Even if there is now a very general existence theorem it is still of importance to study deformation quantizations for such manifolds which carry additional geometric structures. From the whole set of deformation quantization one is looking for one which keeps the additional structure. In this spirit the article deals with the deformation quantization of compact quantizable Kähler manifolds. It was shown 1993 by Bordemann, Meinrenken, and Schlichenmaier [6] that for compact quantizable Kähler manifolds the Berezin-Toeplitz quantization has the correct semi-classical behaviour (see Theorem 2.3 below). Shortly after [6] was submitted we had also the result that by the techniques developed there it was possible to

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construct a deformation quantization [12]. Details were written up in German [32] and the result (with few steps of the proof) appeared in [31, 33]. The complete proof was not published in English.

Compact Kähler manifolds appear as phase spaces of constrained systems and as reduced phase spaces under a group action. More recently, they play a rather prominent role in Chern-Simons theory, topological and 2-dimensional conformal field theory. Here typically, the phase spaces to be quantized are moduli spaces of certain geometric objects. As examples the compactified moduli spaces of stable holomorphic vector bundles (maybe with additional structures) on a Riemann surface show up. The quantum Hilbert spaces appearing in this context are the Verlinde spaces.

Encouraged by the recent interest in deformation quantization evolving in these fields I found it worthwhile to publish the above mentioned results also in English and add some pieces to it. By the construction of the deformation quantization direct relations to the Berezin-Toeplitz quantization, the geometric quantization (via Tuynman's relation), and asymptotic operator representations are given. Hence what is presented here is more than just another existence proof.

In the proof the theory of generalized Toeplitz operators developed by Boutet de Monvel and Guillemin [9, 16] is used in an essential manner. In the article [17] which appeared one year later than [6] it was also explained by Guillemin himself in which way the existence of a deformation quantization follows from the general theory of Toeplitz operators. More precisely, he showed that the subalgebra of operators commuting with the S^1 action (with respect to the sphere bundle Q defined in Section 3) modulo Toeplitz operators of degree $-\infty$ defines via the symbol map a deformation quantization. This follows from results on the principal and subprincipal symbols proved in [9]. Note that essentially the same idea was employed in [6], in the answer to [12], in [31, 32], and will be presented here. Hence, the deformation quantizations obtained will coincide. One might even say that suitable reinterpreted nowadays from the point of view of deformation quantization their existence was implicitly already contained (at least to a certain extend) in the Boutet de Monvel - Guillemin theory of Toeplitz operators.

Only for certain special examples of compact Kähler manifolds direct constructions have been known earlier; see results by Berezin [3], Moreno and Ortega-Navarro [23, 24], and Cahen, Gutt, and Rawnsley [10]. Recently, for all Kähler manifolds (including the noncompact ones) the existence of a deformation quantization with "separation of variables" was shown by Karabegov [18]. Separation of variables says essentially that the deformation quantization "respects" the complex structure. A classification of all such deformation quantizations for a fixed Kähler manifold was also given by Karabegov. Note that his existence proof is on the level of the formal deformation quantization. It does not yield Hilbert spaces and quantum operators like in our approach (which in contrast is restricted to the case of quantizable compact Kähler manifolds). Independently, a similar existence theorem was proven by Bordemann and Waldmann [7] along Fedosov's original approach. Yet another construction was given recently by Reshetikhin and Takhtajan [28].

Finally, let me stress the fact, that the very essential basics of this work go back to joint work with Martin Bordemann and Eckhard Meinrenken. Details have been added by me later on.

The article is organized as follows. In Section 2 the geometric set-up is given and the main result of this article, the theorem on the construction of the deformation quantization (Theorem 2.2) is formulated. The approximation results from [6] are recalled. In Section 3 the necessary details about the Toeplitz structure introduced by Boutet de Monvel and Guillemin are given. They are employed in Section 4 for the construction of the deformation quantization (the star-product), i.e., the proof of Theorem 2.2. In the concluding Section 5 additional properties of the star-product are discussed. It is shown that we have $1 \star g = g \star 1 = g$, i.e., that the star-product is "null on constants" and that it fulfills the parity condition. A trace is constructed. By a result of Tuynman for compact Kähler manifolds the geometric quantization can be expressed in terms of the Berezin-Toeplitz quantization. Using our theorem we see that the geometric quantization yields also a star-product. This star-product is equivalent to the constructed one. The Berezin-Toeplitz star-product will be a local star-product given by bidifferential operators. It will have the property of "separation of variables". This will be shown in [19].

2. The set-up and the main result

Let (M, ω) be a compact (complex) Kähler manifold of complex dimension n. It should be considered as phase space manifold M with symplectic form given by the Kähler form ω. Denote by $C^\infty(M)$ the algebra of (arbitrary often) differentiable functions. Using the Kähler form one assigns to every $f \in C^\infty(M)$ its Hamiltonian vector field X_f and to every pair of functions f and g the Poisson bracket:

$$\omega(X_f, \cdot) = df(\cdot), \qquad \{f, g\} := \omega(X_f, X_g) \ . \tag{1}$$

With the Poisson bracket $C^\infty(M)$ becomes a Poisson algebra.

Assume (M, ω) to be quantizable. This says that there exists an associated quantum line bundle (L, h, ∇) with holomorphic line bundle L, Hermitian metric h on L and connection ∇ compatible with the metric h and the complex structure such that the curvature of the line bundle and the Kähler form ω of the manifold are related as

$$curv_{L,\nabla}(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = -\mathrm{i}\,\omega(X, Y) \ . \tag{2}$$

Equation (2) is called the *quantization condition*. If the metric is represented as a function $\hat{h}$ with respect to local complex coordinates and a local holomorphic frame of the bundle the quantization condition reads as $\mathrm{i}\,\overline{\partial}\partial \log \hat{h} = \omega$.

The quantization condition implies that L is a positive line bundle. By the Kodaira embedding theorem L is ample, which says that a certain tensor power

L^{m_0} of L is very ample, i.e., the global holomorphic sections of L^{m_0} can be used to embed the phase space manifold M into projective space. Note that the embedding is an embedding as complex manifolds not as Kähler manifolds. The embedding dimension is given by the Hirzebruch-Riemann-Roch formula. Hence, quantizable compact Kähler manifolds are as complex manifolds projective algebraic manifolds. The converse is also true, see [31, 2]. In the following we will assume L to be very ample. If L is not very ample we choose $m_0 \in \mathbb{N}$ such that the bundle L^{m_0} is very ample and take this bundle as quantum line bundle and $m_0\omega$ as Kähler form for M. The underlying complex manifold structure is not changed. Please note that for the examples of moduli spaces mentioned in the introduction there is often a natural ample or very ample quantum line bundle.

We take the Liouville measure $\Omega = \frac{1}{n!}\omega^n$ as volume form on M. On the space of C^∞-sections $\Gamma_\infty(M,L)$ we have the scalar product and norm

$$\langle \varphi, \psi \rangle := \int_M h(\varphi, \psi)\, \Omega\,, \qquad ||\varphi|| := \sqrt{\langle \varphi, \varphi \rangle}\,. \tag{3}$$

Let $\mathrm{L}^2(M,L)$ be the L^2-completion of the space of C^∞-sections of the bundle L and $\Gamma_{hol}(M,L)$ be its (due to compactness of M) finite-dimensional closed subspace of holomorphic sections. Let $\Pi : \mathrm{L}^2(M,L) \to \Gamma_{hol}(M,L)$ be the projection.

Definition 2.1 *For $f \in C^\infty(M)$ the Toeplitz operator T_f is defined to be*

$$T_f := \Pi(f\cdot): \quad \Gamma_{hol}(M,L) \to \Gamma_{hol}(M,L)\,. \tag{4}$$

In words: One takes a holomorphic section s and multiplies it with the differentiable function f. The resulting section $f \cdot s$ will only be differentiable. To obtain a holomorphic section one has to project it back on the subspace of holomorphic sections.

The linear map $T : C^\infty(M) \to \mathrm{End}\big(\Gamma_{hol}(M,L)\big)$, $f \to T_f$, is the *Berezin-Toeplitz quantization map*. Because in general $T_f T_g = \Pi(f\cdot)\Pi(g\cdot)\Pi \neq \Pi(fg\cdot)\Pi = T_{fg}$, it is neither a Lie algebra homomorphism nor an associative algebra homomorphism. From the point of view of Berezin's approach [3] the operator T_f has as a contravariant symbol f (see also [34] for relations to Berezin's covariant symbols).

This defines a map from the commutative algebra of functions to a noncommutative finite-dimensional (matrix) algebra. The finite-dimensionality is due to compactness of M. A lot of classical information will get lost. To recover this information one should consider not just the bundle (L, ∇, h) alone but all its tensor powers $(L^m, \nabla^{(m)}, h^{(m)})$ and apply the above constructions for every m. Note that if $\hat{h}$ corresponds to the metric h w.r.t. a holomorphic frame s of the bundle L then $\hat{h}^m$ corresponds to the metric $h^{(m)}$ w.r.t. to the frame $s^{\otimes m}$ for the bundle L^m. In this way one obtains a family of matrix algebras and a family of maps

$$T^{(m)} : C^\infty(M) \to \mathrm{End}\big(\Gamma_{hol}(M, L^m)\big), \qquad f \to T_f^{(m)}\,. \tag{5}$$

This infinite family should in some sense "approximate" the algebra $C^\infty(M)$. (See [5] and the discussion on strict quantization below.) for a definition of such an approximation.) Indeed this family has the correct semi-classical behaviour as is expressed in Theorem 2.3 below.

It also allows to construct a deformation quantization. A deformation quantization is given by a *star-product*. I will use both terms interchangeable. To fix the notation and the factors of i let me recall the definition of a star-product. Let $\mathcal{A} = C^\infty(M)[[\nu]]$ be the algebra of formal power series in the variable ν over the algebra $C^\infty(M)$. A product $\star$ on $\mathcal{A}$ is called a (formal) star-product if it is an associative $\mathbb{C}[[\nu]]$-linear product such that

1. $\mathcal{A}/\nu\mathcal{A} \cong C^\infty(M)$, i.e., $f \star g \bmod \nu = f \cdot g$,
2. $\frac{1}{\nu}(f \star g - g \star f) \bmod \nu = -\mathrm{i}\{f,g\}$,

where $f,g \in C^\infty(M)$. We can also write

$$f \star g = \sum_{j=0}^{\infty} C_j(f,g)\nu^j \ , \tag{6}$$

with $C_j(f,g) \in C^\infty(M)$. The C_j should be $\mathbb{C}$-bilinear in f and g. Conditions 1 and 2 can be reformulated as

$$C_0(f,g) = f \cdot g, \qquad \text{and} \qquad C_1(f,g) - C_1(g,f) = -\mathrm{i}\{f,g\} \ . \tag{7}$$

The aim of this article is to show the following

Theorem 2.2 *There exists a unique (formal) star-product on $C^\infty(M)$*

$$f \star g := \sum_{j=0}^{\infty} \nu^j C_j(f,g), \quad C_j(f,g) \in C^\infty(M), \tag{8}$$

in such a way that for $f,g \in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have with suitable constants $K_N(f,g)$ for all m

$$||T_f^{(m)} T_g^{(m)} - \sum_{0 \le j < N} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)}|| = K_N(f,g) \left(\frac{1}{m}\right)^N \ . \tag{9}$$

This theorem has been proven immediately after [6] was finished. It has been announced in [31, 33] and the proof was written up in German in [32]. In Section 4 I will supply the proof.

Instead of writing (9) we will sometimes use the more intuitive notation

$$T_f^{(m)} \cdot T_g^{(m)} \quad \sim \quad \sum_{j=0}^{\infty} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)} \qquad (m \to \infty) \ . \tag{10}$$

The asymptotics should always be understood in the above precise sense.

In the proof the results expressed in the following theorem are needed. Denote by $||f||_\infty$ the sup-norm of f on M and by $||T_f^{(m)}|| = \sup_{s\in\Gamma_{hol}(M,L^m),s\neq 0} \frac{||T_f^{(m)}s||}{||s||}$ the operator norm on $\Gamma_{hol}(M,L^m)$.

Theorem 2.3. (Bordemann, Meinrenken, Schlichenmaier)
(a) For every $f\in C^\infty(M)$ *there exists* $C>0$ *such that*

$$||f||_\infty - \frac{C}{m} \le ||T_f^{(m)}|| \le ||f||_\infty \,. \tag{11}$$

In particular, $\lim_{m\to\infty} ||T_f^{(m)}|| = ||f||_\infty$.
(b) For every $f,g\in C^\infty(M)$

$$||m\,\mathrm{i}[T_f^{(m)},T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| \quad = \quad O(\frac{1}{m}) \quad \text{as} \quad m\to\infty\,. \tag{12}$$

(c) For every $f,g\in C^\infty(M)$

$$||T_f^{(m)}T_g^{(m)} - T_{f\cdot g}^{(m)}|| \quad = \quad O(\frac{1}{m}) \quad \textit{as} \quad m\to\infty\,. \tag{13}$$

These results are contained in Theorem 4.1, 4.2, resp. in Section 5 in [6]. Note that part (c) also follows from (9) for $N=1$ and generalizes trivially to finitely many functions.

Our result does not prove a strict deformation quantization in the sense of Rieffel [29]. But it is a *strict quantization* (see for the definition [22, 30]). Let $I := \{\frac{1}{m} \mid m\in\mathbb{N}\}\cup\{0\}$ be the topological space with topology coming from the real line. It has 0 as accumulation point. To every $\hbar\in I$, $\hbar\neq 0$, i.e $\hbar = 1/m$, one assigns the algebra $A_{1/m} := \mathrm{End}(\Gamma_{hol}(M,L^m))$ with $||.||_{1/m}$ the operator norm. To 0 one assigns the algebra $A_0 := C^\infty(M)$ with norm $||.||_0 = |.|_\infty$. The map $\hbar\to T_f^{(1/\hbar)}$, with $T_f^{(\infty)} := f$ defines by Theorem 2.3 a continuous field of C^*-algebras on the family $(A_\hbar)_{\hbar\in I}$. From (45) follows that T respects conjugation. By (12) the additional condition for a strict quantization is also fulfilled. Due to the compactness of M the maps $T_f^{(1/\hbar)}$ for $\hbar\neq 0$ are never injective. Hence the strict quantization is not faithful at a fixed level $\hbar$, only in the limit $\hbar\to\infty$.

In [5] and [6] the notion of L_α, resp. $gl(N)$, resp. $su(N)$ quasi-limit was used for this concept. It was conjectured in [5] that for every compact Kähler manifold the Poisson algebra of function is a $gl(N)$ quasi-limit. This was proven in [6]. This result is of special interest in the theory of membranes.

There is another geometric concept of quantization, the *geometric quantization* introduced by Kostant and Souriau. But for compact Kähler manifolds due to Tuynman [35] (see also [5] for a coordinate independent proof) they have the same semi-classical behaviour

$$Q_f^{(m)} = \mathrm{i}\cdot T_{f-\frac{1}{2m}\Delta f}^{(m)}\,. \tag{14}$$

Here $Q_f^{(m)}$ is the well-known operator of geometric quantization (with respect to the quantum line bundle L^m) corresponding to the prequantum operator $P_f^{(m)} = -\nabla^{(m)}_{X_f^{(m)}} + \mathrm{i} f \cdot id$ and Kähler polarization. Kähler polarization means $Q_f^{(m)} = \Pi^{(m)} P_f^{(m)} \Pi^{(m)}$ with the projectors

$$\Pi^{(m)} : \mathrm{L}^2(M, L^m) \to \Gamma_{hol}(M, L^m) \,. \tag{15}$$

In (14), Δ is the Laplacian with respect to the Kähler metric given by ω. In Section 5 I will show that this allows to define a deformation quantization via the operators of geometric quantization. It will be equivalent to the Berezin-Toeplitz deformation quantization.

3. Toeplitz structure

In [6] the set-up for the proof of the approximation results was given. Here I use the same setting. Let me recall for further reference the main definitions. A more detailed exposition can be found in [32]. Take $(U,k) := (L^*, h^{-1})$ the dual of the quantum line bundle, Q the unit circle bundle inside U (with respect to the metric k) and $\tau : Q \to M$ the projection. Note that for the projective space with quantum line bundle the hyperplane section bundle H, the bundle U is just the tautological bundle. Its fibre over the point $z \in \mathbb{P}^N(\mathbb{C})$ consists of the line in $\mathbb{C}^{N+1}$ which is represented by z. In particular, for the projective space the total space of U with the zero section removed can be identified with $\mathbb{C}^{N+1} \setminus \{0\}$. The same picture remains true for the via the very ample quantum line bundle in projective space embedded manifold M. The quantum line bundle will be the pull-back of H (i.e., its restriction to the embedded manifold) and its dual is the pull-back of the tautological bundle.

In the following we use $E \setminus 0$ to denote the total space of the vector bundle E with the image of the zero section removed. Starting from the function $\hat{k}(\lambda) := k(\lambda, \lambda)$ on U we define $\tilde{a} := \frac{1}{2\mathrm{i}}(\partial - \overline{\partial}) \log \hat{k}$ on $U \setminus 0$ (with respect to the complex structure on U) and denote by α its restriction to Q. Now $d\alpha = \tau^* \omega$ (with $d = d_Q$) and $\mu = \frac{1}{2\pi} \tau^* \Omega \wedge \alpha$ is a volume form on Q. With respect to this form we take the L^2-completion $\mathrm{L}^2(Q, \mu)$ of the space of functions on Q. The generalized Hardy space $\mathscr{H}$ is the closure of the functions in $\mathrm{L}^2(Q, \mu)$ which can be extended to holomorphic functions on the whole disc bundle. The generalized Szegö projector is the projection

$$\Pi : \mathrm{L}^2(Q, \mu) \to \mathscr{H} \,. \tag{16}$$

By the natural circle action Q is a S^1-bundle and the tensor powers of U can be viewed as associated bundles. The space $\mathscr{H}$ is preserved by this action. It can be decomposed into eigenspaces $\mathscr{H} = \prod_{m=0}^{\infty} \mathscr{H}^{(m)}$ where $c \in S^1$ acts on $\mathscr{H}^{(m)}$ as multiplication by c^m. Sections of $L^m = U^{-m}$ can be identified with functions ϕ on Q which satisfy the equivariance condition $\phi(c\lambda) = c^m \phi(\lambda)$. It turns out that this identification is an isometry. Recall that $L^2(M,L^m)$ has a scalar product given in an corresponding way to (3). Restricted to the holomorphic objects we obtain an isometry

$$\Gamma_{hol}(M,L^m) \cong \mathcal{H}^{(m)} . \tag{17}$$

There is the notion of Toeplitz structure (Π,Σ) as developed by Boutet de Monvel and Guillemin in [9, 16]. What is needed from there are only the following facts. Π is the Szegö projector (16). The second object is the submanifold

$$\Sigma = \{\, t\alpha(\lambda) \mid \lambda \in Q, t>0 \,\} \ \subset \ T^*Q\setminus 0 \tag{18}$$

of the tangent bundle of Q defined with the help of the 1-form α. They showed that it is a symplectic submanifold. A (generalized) Toeplitz operator of order k is an operator $A:\mathcal{H}\to\mathcal{H}$ of the form $A=\Pi\cdot R\cdot\Pi$ where R is a pseudodifferential operator (ΨDO) of order k on Q. The Toeplitz operators build a ring. The symbol of A is the restriction of the principal symbol of R (which lives on T^*Q) to Σ. Note that R is not fixed by A, but Guillemin and Boutet de Monvel showed that the symbols are well-defined and that they obey the same rules as the symbols of ΨDOs. In particular we have the following relations

$$\sigma(A_1A_2) = \sigma(A_1)\sigma(A_2), \qquad \sigma([A_1,A_2]) = \mathrm{i}\{\sigma(A_1),\sigma(A_2)\}_\Sigma. \tag{19}$$

In our context only two Toeplitz operators appear:

(1) The generator of the circle action gives the operator $D_\varphi = \frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}$. It is an operator of order 1 with symbol t. It operates on $\mathcal{H}^{(m)}$ as multiplication by m.

(2) For $f\in C^\infty(M)$ let M_f be the operator on $L^2(Q,\mu)$ corresponding to multiplication with τ^*f. We set [1] $T_f=\Pi\cdot M_f\cdot\Pi:\mathcal{H}\to\mathcal{H}$. Because M_f is constant along the fibres of τ, T_f commutes with the circle action. Hence $T_f=\prod_{m=0}^{\infty} T_f^{(m)}$, where $T_f^{(m)}$ denotes the restriction of T_f to $\mathcal{H}^{(m)}$. After the identification of $\mathcal{H}^{(m)}$ with $\Gamma_{hol}(M,L^m)$ we see that these $T_f^{(m)}$ are exactly the Toeplitz operators $T_f^{(m)}$ introduced in Section 2. In this sense T_f is called the global Toeplitz operator and the $T_f^{(m)}$ the local Toeplitz operators. T_f is an operator of order 0. Let us denote by $\tau_\Sigma:\Sigma\subseteq T^*Q\to Q\to M$ the composition then we obtain for the symbol $\sigma(T_f)=\tau_\Sigma^*(f)$.

4. Proof of Theorem 2.2

Let the notation be as in the last section. In particular, let T_f be the Toeplitz operator, D_φ the operator of rotation, and $T_f^{(m)}$, resp. $(m\cdot)$ their projections on the eigenspaces $\mathcal{H}^{(m)}\cong\Gamma_{hol}(M,L^m)$.

---

[1] There should be no confusion with the operator $T_f = T_f^{(1)}$ introduced above.

4.1. The definition of the $C_j(f,g) \in C^\infty(M)$

The construction is done inductively in such a way that

$$A_N = D_\varphi^N T_f T_g - \sum_{j=0}^{N-1} D_\varphi^{N-j} T_{C_j(f,g)} \tag{20}$$

is always a Toeplitz operator of order zero. The operator A_N is S^1-invariant, i.e., $D_\varphi \cdot A_N = A_N \cdot D_\varphi$. Because it is of order zero his symbol is a function on Q. By the S^1-invariance the symbol is even given by (the pull-back of) a function on M. Take this function to be the next element $C_N(f,g)$ in the star-product. By construction the operator $A_N - T_{C_N(f,g)}$ is of order -1 and $A_{N+1} = D_\varphi(A_N - T_{C_N(f,g)})$ is of order 0 and hence exactly of the form given in (20). The induction starts with

$$A_0 = T_f T_g, \qquad \text{and} \tag{21}$$

$$\sigma(A_0) = \sigma(T_f)\sigma(T_g) = \tau_\Sigma^*(f) \cdot \tau_\Sigma^*(g) = \tau_\Sigma^*(f \cdot g)\ . \tag{22}$$

Hence, $C_0(f,g) = f \cdot g$ as required.

It remains to show statement (9) about the asymptotics. As an operator of order zero on a compact manifold A_N is bounded. Hence the same is true for all its restrictions $A_N^{(m)}$ to $\mathscr{H}^{(m)}$. If we calculate them we obtain

$$||m^N T_f^{(m)} T_g^{(m)} - \sum_{j=0}^{N-1} m^{N-j} T_{C_j(f,g)}^{(m)}|| = ||A_N^{(m)}|| \le ||A_N||\ . \tag{23}$$

After dividing by m^N Equation (9) follows. Bilinearity is clear.

4.2. The Poisson structure

The relation $C_0(f,g) = f \cdot g$ was proven above. To show the 2. formula in (7) we write explicitly (23) for $N = 2$ and the pair of functions (f,g):

$$||m^2 T_f^{(m)} T_g^{(m)} - m^2 T_{f \cdot g}^{(m)} - m T_{C_1(f,g)}^{(m)}|| \le K\ . \tag{24}$$

A corresponding expression is obtained for the pair (g,f). If we subtract both operators inside of the norm we obtain (with the triangle inequality and suitable K')

$$||m^2 (T_f^{(m)} T_g^{(m)} - T_g^{(m)} T_f^{(m)}) - m(T_{C_1(f,g)}^{(m)} - T_{C_1(g,f)}^{(m)})|| \le K'\ . \tag{25}$$

Dividing by m and multiplying with i we obtain

$$||m\mathrm{i}[T_f^{(m)}, T_g^{(m)}] - T_{\mathrm{i}\left(C_1(f,g) - C_1(g,f)\right)}^{(m)}|| = O(\frac{1}{m})\ . \tag{26}$$

Using the asymptotics given by Theorem 2.3 (b) for the commutator we get

$$||T_{\{f,g\} - \mathrm{i}\left(C_1(f,g) - C_1(g,f)\right)}^{(m)}|| = O(\frac{1}{m})\ . \tag{27}$$

Taking the limit for $m \to \infty$ and using Theorem 2.3 (a) we get

$$||\{f,g\} - \mathrm{i}(C_1(f,g) - C_1(g,f))||_\infty = 0 \,. \tag{28}$$

Hence $\{f,g\} = \mathrm{i}(C_1(f,g) - C_1(g,f))$. This shows (7).

4.3. The uniqueness

It is proven by induction using the asymptotics (9). Let $C_j(f,g)$ and $\tilde{C}_j(f,g)$ be two such systems of bilinear maps fulfilling the required properties. Assume $C_j = \tilde{C}_j$ for $j \le N-2$. If we subtract the corresponding expressions in (9) and use the fact that $T^{(m)}$ is linear we obtain

$$||\frac{1}{m^{N-1}} T^{(m)}_{(C_{N-1}(f,g) - \tilde{C}_{N-1}(f,g))}|| \le \frac{K}{m^N} \,. \tag{29}$$

Hence,

$$\lim_{m\to\infty} ||T^{(m)}_{(C_{N-1}(f,g) - \tilde{C}_{N-1}(f,g))}|| = 0 \,. \tag{30}$$

With Theorem 2.3 (a) it follows $C_{N-1}(f,g) = \tilde{C}_{N-1}(f,g)$. The induction starts with $N = 1$. But here $C_0(f,g) = \tilde{C}_0(f,g) = f \cdot g$ is required.

4.4. The associativity

The proof employs the associativity of the operators used to construct the star-product and again Theorem 2.3 (a). The relation $f \star (g \star h) = (f \star g) \star h$ can be rewritten in relations for the maps C_j:

$$\sum_{l=0}^{k} C_l(f, C_{k-l}(g,h)) = \sum_{l=0}^{k} C_l(C_{k-l}(f,g),h) \,. \tag{31}$$

From Theorem 2.3 (a) we know $f = g \longleftrightarrow \lim_{m\to\infty} ||T_f^{(m)} - T_g^{(m)}|| = 0$. Hence it is enough to apply the Toeplitz operator $T^{(m)}$ to the relation (31) and study the asymptotics of $T_{\text{left-hand side}} - T_{\text{right-hand side}}$. This is done by induction over k.
$k = 0: C_0(f, C_0(g,h)) = C_0(C_0(f,g),h)$ is true because $C_0(f,g) = f \cdot g$.
Assume the claim to be true up to level $k-1$. The equation (9) for $0 \le r \le k$ multiplied by m^r $(N = r+1)$ yields

$$T^{(m)}_{C_r(f,g)} = m^r T_f^{(m)} T_g^{(m)} - \sum_{s=0}^{r-1} m^{r-s} T_{C_s(f,g)} + O(\frac{1}{m}) \,. \tag{32}$$

Here the symbol $O(\frac{1}{m})$ is shorthand for the statement that the difference of the operators on the left and on the right is an operator whose norm behaves like $O(\frac{1}{m})$ for $m \to \infty$. In particular we obtain for $l = 0, 1, \ldots, k$

$$T^{(m)}_{C_l(f, C_{k-l}(g,h))} = m^l T_f^{(m)} T^{(m)}_{C_{k-l}(g,h)} - \sum_{s=0}^{l-1} m^{l-s} T_{C_s(f, C_{k-l}(g,h))} + O(\frac{1}{m}) \,. \tag{33}$$

Summation over l yields

$$T^{(m)}_{\text{l.h.s.}} = \sum_{l=0}^{k} m^l T_f^{(m)} T^{(m)}_{C_{k-l}(g,h)} - \sum_{l=0}^{k} \sum_{s=0}^{l-1} m^{l-s} T^{(m)}_{C_s(f,C_{k-l}(g,h))} + O(\frac{1}{m}) \,. \tag{34}$$

The second sum can be rewritten as

$$-\sum_{r=1}^{k} m^r \sum_{l=r}^{k} T^{(m)}_{C_{l-r}(f,C_{k-l}(g,h))} = -\sum_{r=1}^{k} m^r T^{(m)}_{\sum_{s=0}^{k-r} C_s(f,C_{k-r-s}(g,h))} \,. \tag{35}$$

For such sums we know by induction that (31) is valid. The same is done for the right hand side. If we subtract $T^{(m)}_{\text{r.h.s.}}$ from $T^{(m)}_{\text{l.h.s.}}$, it remains

$$\sum_{l=0}^{k} m^l T_f^{(m)} T^{(m)}_{C_{k-l}(g,h)} - \sum_{l=0}^{k} m^l T^{(m)}_{C_{k-l}(f,g)} T_h^{(m)} + O(\frac{1}{m}) \,. \tag{36}$$

By splitting the first sum into $l = 0$ and $l \geq 1$ and using for the $l = 0$ term the asymptotic (32) we obtain

$$m^0 \left(T_f^{(m)} m^k T_g^{(m)} T_h^{(m)} - \sum_{s=0}^{k-1} m^{k-s} T_f^{(m)} T^{(m)}_{C_s(g,h)} + O(\frac{1}{m}) \right) + \sum_{l=1}^{k} m^l T_f^{(m)} T^{(m)}_{C_{k-l}(g,h)} \tag{37}$$

$$= m^k T_f^{(m)} (T_g^{(m)} T_h^{(m)}) + O(\frac{1}{m}) \,. \tag{38}$$

A corresponding expression follows for the second sum. As difference remains

$$m^k (T_f^{(m)} (T_g^{(m)} T_h^{(m)}) - (T_f^{(m)} T_g^{(m)}) T_h^{(m)}) + O(\frac{1}{m}) \,. \tag{39}$$

Now we ended up with operators which are clearly associative, The operator coming with the m^k term vanishes. Hence associativity follows from Theorem 2.3 (a). ■

5. Additional properties

The introduced star-product has important properties.

5.1. Unit

The unit of the algebra $C^\infty(M)$, the constant function 1, will also be the unit in the star-product. Such star-products are sometimes called to have the property "null on constants".

Proposition 5.1 *For the above introduced star-product we have*

$$1 \star g = g \star 1 = g \,. \tag{40}$$

Equivalently,

$$C_k(1,g) = C_k(g,1) = 0, \qquad \text{for} \quad k \geq 1 \,. \tag{41}$$

Proof. For $f \equiv 1$ we have $T_f \equiv id$, resp. $T_f^{(m)} \equiv id$. Also $C_0(1,g) = g = C_0(g,1)$. Further with (20)

$$A_1 = D_\varphi T_f T_g - D_\varphi T_{fg} = D_\varphi T_g - D_\varphi T_g \, . \tag{42}$$

Hence the symbol of A_1 vanishes. But this implies $C_1(1,g) = 0 = C_1(g,1)$. The claim follows by trivial induction from (20). ■

5.2. Parity

A star-product is said to fulfill the parity condition if

$$\overline{f \star g} = \overline{g} \star \overline{f} \, . \tag{43}$$

Considering the formal parameter to be real ($\overline{\nu} = \nu$) this is equivalent to

$$\overline{C_k(f,g)} = C_k(\overline{g}, \overline{f}), \quad k \geq 0 \, . \tag{44}$$

We will show

Proposition 5.2 *The above introduced star-product fulfills parity.*

Lemma 5.3

$$T_f^{(m)*} = T_{\overline{f}}^{(m)} \, . \tag{45}$$

Proof. Take any $s, t \in \Gamma_{hol}(M, L^m)$. For the scalar product we calculate ($\Pi^{(m)}$ is the projector defined in (15))

$$\langle s, T_f^{(m)} t \rangle = \langle s, \Pi^{(m)} f t \rangle = \langle s, f t \rangle = \langle \overline{f} s, t \rangle = \langle T_{\overline{f}}^{(m)} s, t \rangle \, . \tag{46}$$

Hence the claim. ■

Proof. (Proposition 5.2) Recall that the identification of the sections of L^m with equivariant functions on the circle bundle Q is an isomorphy. Hence the definition of adjoint operators agree. For the global Toeplitz operator we obtain $T_f^* = \prod_{m=0}^{\infty} T_{\overline{f}}^{(m)} = T_{\overline{f}}$. The star-product

$$\overline{g} \star \overline{f} = \sum_{j=0}^{\infty} \nu^j C_j(\overline{g}, \overline{f}) \tag{47}$$

is given via the asymptotic expansion of

$$T_{\overline{g}}^{(m)} \cdot T_{\overline{f}}^{(m)} = T_g^{(m)*} \cdot T_f^{(m)*} = (T_f^{(m)} \cdot T_g^{(m)})^* \, . \tag{48}$$

For the asymptotic expansion of the last expression we have

$$(T_f^{(m)} \cdot T_g^{(m)})^* \quad \sim \quad \sum_{j=0}^{\infty} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)*} = \sum_{j=0}^{\infty} \left(\frac{1}{m}\right)^j T_{\overline{C_j(f,g)}}^{(m)} . \tag{49}$$

But this is the complex conjugate of the asymptotic expansion which defines $f \star g$. This shows (43) ■

By the parity condition we have on $C^\infty(M)[[\nu]]$ an anti-involution given by pointwise complex conjugation on the functions, and by considering the formal parameter to be real ($\overline{\nu} = \nu$).

5.3. Locality and separation of variables

Recall that a star-product is local if for all $f, g \in C^\infty(M)$ the support $supp\, C_j(f,g)$ is contained in $supp\, f \cap supp\, g$ for all $j \in \mathbb{N}_0$. Using Peetre's theorem and the fact that the C_j are bilinear this implies that for a local star-product the C_j can be given by bidifferential operators.

One way to proof locality is by studying the symbol calculus of degree zero Toeplitz operators which commute with the S^1 action in more detail (see [17, 9]). Another possibility (which gives a different perspective) is to use the fact that the projection operators $\Pi^{(m)}$ can be expressed with the help of Berezin-Rawnsley's coherent states and the fact that the coherent states are "localizing" for $m \to \infty$. The details will appear in [19].

From the locality it follows that the star-product can be restricted to open subsets and defines compatible star-products there. For such star-products Karabegov introduced the notion of star-products with *separation of variables* [18] (Bordemann and Waldmannn [7] called them star-products of Wick type). In our convention this reads as $f \star k = f \cdot k$ and $k \star g = k \cdot g$ for (locally defined) holomorphic functions g, antiholomorphic functions f and arbitrary functions k. The above introduced star-product will be a star-product with separation of variables.

More precisely, we expect that the C_j are bidifferential operators of degree (j, j) with only holomorphic derivatives in the first entry and only antiholomorphic derivatives in the second entry. See Section 5.5 for examples.

5.4. Trace

Proposition 5.4. (Bordemann, Meinrenken, Schlichenmaier) *Let $f \in C^\infty(M)$ and let $n = \dim_{\mathbb{C}} M$. Denote the trace on* $\mathrm{End}(\Gamma_{hol}(M, L^m))$ *by* $\mathrm{Tr}^{(m)}$ *then*

$$\mathrm{Tr}^{(m)}\,(T_f^{(m)}) = m^n \left(\frac{1}{\mathrm{vol}(\mathbb{P}^n(\mathbb{C}))} \int_M f\,\Omega + O(m^{-1}) \right) . \tag{50}$$

This result can also be found in [6]. There it was given only with a hint of its proof. Because it is central for the following let me give the details.

Proof. Let us start with a real valued f. Then the operator T_f and the components $T_f^{(m)}$ are self adjoint (see (45)). Let $d(m) = \dim \mathscr{H}^{(m)}$ and let $\lambda_1^{(m)}, \lambda_2^{(m)}, \dots, \lambda_{d(m)}^{(m)}$ be the eigenvalues of the restriction of T_f on $\mathscr{H}^{(m)}$. In particular, these are also the eigenvalues of $T_f^{(m)}$ on $\Gamma_{hol}(M, L^m)$. Following [9] ($n = \dim_{\mathbb{C}} M$) let

$$\mu_m = \frac{1}{m^n} \sum_{i=1}^{d(m)} \delta(\lambda - \lambda_i^{(m)}) \tag{51}$$

be the discrete spectral measure. By Theorem 13.13 of [9] it converges weakly to the limit measure

$$\mu(g) = \gamma_M \int_M g(f(z))\,\Omega(z) \tag{52}$$

with a universal constant γ_M only depending on the manifold M. An important intermediate result there is the asymptotic expansion (Equation 13.13 in [9])

$$\mu_m(g) \quad \sim \quad \sum_{r=-n}^{\infty} a_r(g) m^{r+n} \,. \tag{53}$$

For $g \equiv 1$ we obtain

$$\frac{1}{m^n} \sum_{i=1}^{d(m)} \lambda_i^{(m)} = \frac{1}{m^n} \operatorname{Tr}^{(m)} T_f^{(m)} = \gamma_M \int_M f\,\Omega + O(\frac{1}{m}) \,. \tag{54}$$

To calculate γ_M we evaluate (54) for $f \equiv 1$ (i.e., $T_f^{(m)} = id$) and obtain

$$\gamma_M = \frac{\dim \Gamma_{hol}(M, L^m)}{m^n \cdot \operatorname{vol}(M)} + O(\frac{1}{m}) \,. \tag{55}$$

Note that (see p.113 and Thm 5.22 in [25])

$$\dim \Gamma_{hol}(M, L^m) = \frac{\operatorname{vol}(M)}{\operatorname{vol}(\mathbb{P}^n(\mathbb{C}))} \cdot m^n + O(m^{n-1}) \,. \tag{56}$$

Hence $\gamma_M = \operatorname{vol}(\mathbb{P}^n(\mathbb{C}))^{-1}$. In particular the coefficient depends only on the dimension of M. This shows the claim for real valued f. For complex valued f it follows from linearity by considering real and imaginary parts separately. In [25] for M the restriction $\omega_{FS|M}$ of the Fubini-Study Kähler form was used to define the volume. Here we have to work with the form ω. Because the de Rham classes of both forms coincide and the Kähler forms are closed, the volume will be the same. ■

From (53) follows the asymptotic expansion for $m \to \infty$ (see also [8])

$$\operatorname{Tr}^{(m)}(T_f^{(m)}) \quad \sim \quad m^n \left(\sum_{j=0}^{\infty} \left(\frac{1}{m}\right)^j \tau_j(f) \right), \quad \text{with} \quad \tau_j(f) \in \mathbb{C} \,. \tag{57}$$

We define the $\mathbb{C}[[\nu]]$-linear map

$$\operatorname{Tr}: C^\infty(M)[[\nu]] \to \nu^{-n}\mathbb{C}[[\nu]], \quad \operatorname{Tr} f := \nu^{-n}\sum_{j=0}^{\infty} \nu^j \tau_j(f), \tag{58}$$

such that for $f \in C^\infty(M)$ the $\tau_j(f)$ are given by the asymptotic expansion (57) and for arbitrary elements by $\mathbb{C}[[\nu]]$-linear extension.

Proposition 5.5 *The map* Tr *is a trace, i.e., we have*

$$\operatorname{Tr}(f \star g) = \operatorname{Tr}(g \star f)\ . \tag{59}$$

Proof. By $\mathbb{C}[[\nu]]$-linearity it is enough to show this for $f, g \in C^\infty(M)$. The element $f \star g - g \star f$ is given by the asymptotic expansion of $T_f^{(m)} \cdot T_g^{(m)} - T_g^{(m)} \cdot T_f^{(m)}$. Hence $\operatorname{Tr}(f \star g - g \star f)$ is given by the expansion of

$$\operatorname{Tr}^{(m)}(T_f^{(m)} \cdot T_g^{(m)} - T_g^{(m)} \cdot T_f^{(m)})\ . \tag{60}$$

But for every m this vanishes. Hence (59) follows. ■

5.5. Examples

For the sphere S^2, resp. $\mathbb{P}^1(\mathbb{C})$ with Kähler form

$$\omega = \frac{\mathrm{i}}{(1+z\bar{z})^2}\, dz \wedge d\bar{z}\ , \tag{61}$$

and the hyperplane bundle as quantum line-bundle explicit calculations [2] of the author (not published) yield (using $T^{(m)}(f) := T_f^{(m)}$)

$$\lim_{m\to\infty} ||m\left(T^{(m)}(f)T^{(m)}(g) - T^{(m)}(fg)\right) + T^{(m)}\left((1+z\bar{z})^2 \frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}}\right)|| = 0\ . \tag{62}$$

This implies

$$C_1(f,g) = -(1+z\bar{z})^2 \frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}}\ . \tag{63}$$

For the case of Riemann surfaces of genus $g \geq 2$ more than half of the article [20] by Klimek and Lesniewski deals with the proof of the fact corresponding to (62). In the realization of the Riemann surface M as quotient space $\{z \in \mathbb{C} \mid |z| < 1\}/G$ with G a Fuchsian subgroup of $\mathrm{SU}(1,1)$ acting by fractional linear transformations one takes as Kähler form the $\mathrm{SU}(1,1)$ invariant form

$$\omega = \frac{2\mathrm{i}}{(1-z\bar{z})^2} dz \wedge d\bar{z}\ . \tag{64}$$

---

[2] Not following the lines of the proof in Section 4 but working with a basis of the sections of the bundles.

The corresponding quantum line bundle is the canonical line bundle, i.e., the bundle whose local sections are the holomorphic differentials. From their results follows

$$C_1(f,g) = -\frac{1}{2}(1-z\bar{z})^2 \frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} . \tag{65}$$

5.6. Deformation quantization via geometric quantization

Via Tuynman's relation (14) the operator $Q^{(m)}$ of geometric quantization corresponding to the function f can be expressed by the Toeplitz operator $T^{(m)}$ corresponding to the function $f - \frac{1}{2m}\Delta f$. Theorem 2.2 shows that the asymptotic expansion

$$Q_f^{(m)} \cdot Q_g^{(m)} \quad \sim \quad \sum_{j=0}^{\infty} \left(\frac{1}{m}\right)^j Q^{(m)}_{D_j(f,g)} , \tag{66}$$

with suitable $D_j(f,g) \in C^\infty(M)$ is well-defined in the precise sense as expressed in the theorem there. We set

$$f \star_G g := \sum_{j=0}^{\infty} \nu^j D_j(f,g) . \tag{67}$$

The first two terms calculate as

$$D_0(f,g) = f \cdot g \tag{68}$$

$$D_1(f,g) = C_1(f,g) + \frac{1}{2}(\Delta(f\cdot g) - \Delta f \cdot g - f \cdot \Delta g), \tag{69}$$

where the C_j are the coefficients of the Berezin-Toeplitz star-product. In particular the conditions (7) are fulfilled for the D's. Hence, this defines indeed a star-product.

In fact more is valid. If we introduce the linear maps

$$B^{(m)}(f) := f - \frac{1}{2m}\Delta f, \tag{70}$$

and the $\mathbb{C}[[\nu]]$-linear map induced by

$$B(f) := f - \nu\frac{\Delta}{2} f = (id - \nu\frac{\Delta}{2})f \tag{71}$$

on $C^\infty(M)[[\nu]]$ we can rewrite (66)

$$T^{(m)}_{B^{(m)}(f)} \cdot T^{(m)}_{B^{(m)}(g)} \quad \sim \quad \sum_{j=0}^{\infty} \left(\frac{1}{m}\right)^j T^{(m)}_{B^{(m)}(D_j(f,g))} . \tag{72}$$

Taking the asymptotics we get $B(f) \star B(g) = B(f \star_G g)$. Note that $B(f) \mod \nu = f$, $B(1) = 1$, and that B is invertible. The inverse is given by

$$B^{-1} = id + \sum_{k=1}^{\infty} \frac{1}{2^k} \nu^k \Delta^k . \tag{73}$$

Recall that two star-products (over the same manifold) are equivalent if there exists a $\mathbb{C}[[\nu]]$-algebra isomorphism inducing the identity on the zero order part. This implies

Proposition 5.6 *The star-product of geometric quantization is equivalent to the star-product of Berezin-Toeplitz quantization.*

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Homotopy Gerstenhaber algebras

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Abstract. The purpose of this paper is to complete Getzler-Jones' proof of Deligne's Conjecture, thereby establishing an explicit relationship between the geometry of configurations of points in the plane and the Hochschild complex of an associative algebra. More concretely, it is shown that the B_∞-operad, which is generated by multilinear operations known to act on the Hochschild complex, is a quotient of a certain operad associated to the compactified configuration spaces. Different notions of homotopy Gerstenhaber algebras are discussed: One of them is a B_∞-algebra, another, called a homotopy G-algebra, is a particular case of a B_∞-algebra, the others, a G_∞-algebra, an $\overline{E}^1$-algebra, and a weak G_∞-algebra, arise from the geometry of configuration spaces. Corrections to the paper of Kimura, Zuckerman, and the author related to the use of a nonextant notion of a homotopy Gerstenhaber algebra are made.

Keywords: Gerstenhaber algebra, Hochschild cohomology, homotopy algebra, little disks operad, moduli space of punctured spheres

Mathematics Subject Classifications (2000): 16E40, 18C99, 55P99, 14H10, 17A30, 17A42

Dedicated to the memory of Moshé Flato

In an unpublished paper of E. Getzler and J. D. S. Jones [10], the notion of a homotopy n-algebra was introduced. Unfortunately the construction that justified the definition contained an error, which passed unnoticed in subsequent work, in spite of being heavily used in it. That work included the solution by Getzler and Jones [10] of Deligne's Conjecture, whose weak version had been proven in [28]; the construction by T. Kimura, G. Zuckerman, and the author [21] of a homotopy Gerstenhaber algebra structure (called a G_∞-algebra therein) on the state space of a topological conformal field theory (TCTF); the extensions of the above work by Akman [1, 2] and Gerstenhaber and the author [28]; a few papers delivered at the Workshop on Operads in Osnabrück in June 1998 [29]. The purpose of this paper is to correct the error in the original construction of [10], complete Getzler-Jones' proof of Deligne's Conjecture accordingly, and make appropriate corrections in [21].

First, let us describe the problem. A *Gerstenhaber (G-) algebra* is defined by two operations, a (dot) product ab and a bracket $[a,b]$, on a graded vector space V over a ground field k of characteristic zero, so that the product defines a graded commutative algebra structure on V and the bracket a graded Lie algebra structure on $V[1]$, the desuspension of the graded vector space $V = \bigoplus_n V^n$: $V[1]^n = V^{n+1}$.

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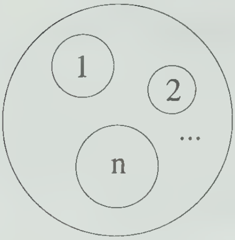

Figure 1. The little disks operad.

The bracket must be a graded derivation of the product in the following sense:

$$[a,bc] = [a,b]c + (-1)^{(|a|-1)|b|} b[a,c],$$

where $|a|$ denotes the degree of an element $a \in V$. In other words, a G-algebra is a specific graded version of a Poisson algebra.

A G-algebra may be equivalently defined as an algebra over the operad $e_2 = H_\bullet(D) = H_\bullet(D;k)$ of the homology of the *little disks operad.* This is a collection of manifolds $D(n)$, $n \geq 1$, where each $D(n)$ is the configuration space of n nonoverlapping little disks inside the standard unit disk in the plane, see Figure 1.

The space $D(n)$ is obviously an open set in $\mathbb{R}^{3n}$ (whence a manifold structure), each configuration being uniquely determined by the position of the centers of the disks and their radii. It is assumed that each little disk is labeled by a number from 1 through n, which defines the action of the permutation group Σ_n on $D(n)$. The operad composition

$$\gamma : D(k) \times D(n_1) \times \cdots \times D(n_k) \to D(n_1 + \cdots + n_k)$$

is given by scaling down given configurations in $D(n_1), \ldots, D(n_k)$, gluing them into the k holes in a given configuration in $D(k)$, and erasing the seams. Thus $D = \{D(n) \mid n \geq 1\}$ becomes an operad of manifolds.

The way the little disks operad D has relevance to G-algebras is through the following theorem.

Theorem 0.1. (F. Cohen [5]) *The structure of a G-algebra on a $\mathbb{Z}$-graded vector space is equivalent to the structure of an algebra over the homology little disks operad $H_\bullet(D)$.*

In view of this theorem, we will refer to the operad $e_2 := H_\bullet(D)$ as the *G-operad.*

On the other hand, a purely algebraic example of a G-algebra was given by the following result.

Theorem 0.2. (M. Gerstenhaber [8]) *The Hochschild cohomology of an associative algebra has the natural structure of a G-algebra with respect to the cup product and the Gerstenhaber (G-) bracket.*

Thus the Hochschild cohomology is naturally an algebra over the homology little disks operad e_2. In a letter [6], P. Deligne pointed out that there must be more than this formal relationship between the little disks operad and the Hochschild cohomology, which later became quoted as the following conjecture.

Conjecture 0.3. (Deligne's Conjecture) *The structure of an algebra over the homology little disks operad e_2 on the Hochschild cohomology may be naturally lifted to the (co)chain level.*

The goal of this paper is to prove a version of this conjecture, see Corollary 3.2. The conjecture has found several interpretations, both algebraic and geometric. Algebraically, the conjecture would imply the structure of a homotopy G-algebra on the Hochschild cochain complex. Here is an account of what is known about it up to date.

- A homotopy G-algebra structure on the Hochschild complex was defined by Gerstenhaber and the author [28]. Apart from the cup product and the G-bracket, we used higher operations (called *braces*) on the Hochschild complex constructed by T. Kadeishvili [14] and Getzler [9], and wrote down a set of identities for the operations interpreted as homotopies for the G-algebra identities and a hierarchy of homotopies between homotopies. The bilinear and trilinear braces providing homotopies for the commutativity of the cup product and the distributivity of the G-bracket, respectively, had been known since the original paper of Gerstenhaber [8].

- Getzler and Jones [10] showed that the same operations on the Hochschild complex defined the structure of a *B_∞-algebra*, a more general version of a homotopy G-algebra introduced by H. Baues in the study of the double bar construction in algebraic topology, see Definition 2.1 below. The B_∞-algebra structure on the Hochschild complex is obtained by setting one of the B_∞-operations to be the dot product, some others to be the braces, and the others to zero. In terms of operads, the B_∞-structure on the Hochschild complex is obtained by realizing the homotopy G-operad of the previous paragraph as a quotient of the B_∞-operad and using the homotopy G-structure on the Hochschild complex.

- Tamarkin [26] extended the cup product and the G-bracket on the Hochschild complex to the structure of a *G_∞-algebra*, which is the most canonical notion of a homotopy G-algebra. It may be defined in terms of the corresponding operad, which is the minimal model of the G-operad in the sense of M. Markl [23]. The G_∞-algebra structure of Tamarkin was again constructed by defining a morphism of operads $G_\infty \to B_\infty$ and using the B_∞-algebra structure described in the previous two paragraphs. The construction of the morphism $G_\infty \to B_\infty$ is very involved: It uses the existence of Drinfeld's associator and Etingof-Kazhdan's quantization theorem. Tamarkin [26] used this construction in his algebraic proof of Kontsevich's Formality Theorem [17], which implied the Deformation Quantization Conjecture [4].

Remark 1 Note that we used the same notation "G_∞-algebra" for a different object in our paper [21]. Since as it has turned out, that object does not exist, this should not create any confusion.

The above results are purely algebraic. However they bring more evidence to the relationship between the geometry of the little disks operad and the algebra of the Hochschild complex hinted by Deligne's Conjecture. Moreover, to establish this relationship, one just needs to relate the above operads (the homotopy G-operad, B_∞, and G_∞) to the little disks operad. Here is what is known in this direction.

- Getzler and Jones [10] noticed that the G_∞-operad is isomorphic to the the first term of a spectral sequence associated to the so-called "topological" filtration of the moduli space operad of configurations of points in the plane. This operad is homotopy equivalent to the little disks operad. For the moduli space operad, Getzler and Jones also offered the geometric construction of a cellular model which mapped surjectively to the B_∞-operad. Unfortunately, there was an error in the construction: The "cellular model" was, strictly speaking, not cellular, because its components were not cell complexes. A counterexample was found by Tamarkin.

- Another interesting relation between the G_∞-operad and the little disks operad is the idea of a recent preprint of Tamarkin [27], who showed that the singular chain operad $C_\bullet(D)$ of little disks is *formal*, i.e., quasi-isomorphic to its homology $H_\bullet(D)$. Kontsevich [18] found a very simple geometric proof of this result. Both results imply that there exists a morphism $G_\infty \to C_\bullet(D)$ unique up to homotopy.

Since all the algebraic solutions of Deligne's Conjecture so far used the explicit B_∞-structure on the Hochschild complex, finding an explicit relationship between the B_∞-operad and the little disks operad seems to be most crucial for understanding the relationship between algebra and geometry suggested by Deligne's Conjecture. Moreover, the operads G_∞ and $C_\bullet(D)$, which have obvious geometric meaning, do not act explicitly on the Hochschild complex. In our understanding, placing the B_∞-operad within the topology of the little disks operad will provide the most complete solution of Deligne's Conjecture.

In this paper, we show that the B_∞-operad is a quotient of a certain operad $\overline{E}^1$ arising geometrically from the moduli space operad of configurations of points in the plane. The operad $\overline{E}^1$ is the first term of the spectral sequence associated to a filtration of the moduli space operad. We show that the operad $\overline{E}^1$ is free as an operad of graded vector spaces and quasi-isomorphic to the G-operad. We construct an explicit surjection from the operad $\overline{E}^1$ to the B_∞-operad, which will imply an explicit $\overline{E}^1$-algebra structure on the Hochschild complex. We find it truly remarkable that the object B_∞ of the algebraic world (with its natural action on the Hochschild complex) is ruled by the geometry of the configuration spaces (through the operad $\overline{E}^1$). For example, see Section 2.3, where the defining relations of a B_∞-algebra are read off from the incidence relations for the strata in the moduli spaces.

Since $\overline{E}^1$ is quasi-isomorphic to the G-operad, it follows from homotopy theory of operads that there exists a morphism $G_\infty \to \overline{E}^1$. This also implies the existence of a G_∞-algebra structure on the Hochschild complex.

At the end we will indicate which changes are to be made to the paper [21] by Kimura, Zuckerman, and the author, which utilized the incorrect notion of a G_∞-algebra. Briefly, the changes are that this notion should be replaced by the correct one, which brings corrections to the identities described implicitly in [21]. However, all the identities written out in [21] explicitly do not require corrections.

Remark 2 While this paper was in preparation, there were made two announcements of results of similar nature. J. McClure and J. Smith [24] constructed a cellular operad acting on the Hochschild complex and announced that it was homotopy equivalent to the little disks operad. Kontsevich [18] announced the construction of an operad with an explicit quasi-isomorphism to the G-operad and an explicit action on the Hochschild complex, along with the proof of his multi-dimensional generalization of Deligne's Conjecture. The proof of Deligne's Conjecture along these lines is published in this volume, see [19].

We will use the following *terminology* regarding basic notions of topology. All *topological spaces* considered will be Hausdorff, except when referring to complex algebraic curves, we will use standard terminology of Zariski topology, such as an irreducible component. For a topological space X, let $X^\bullet$ denote its *one-point compactification*, which is $X^\bullet = X$ if X is compact and $X^\bullet = X \cup \{\infty\}$ otherwise. A (*p-dimensional*) *cell* in a topological space X is a subset $E \subset X$ along with a continuous map $f : I^p \to \bar{E}$, where I^p is the closed unit cube in $\mathbb{R}^p$ with the boundary ∂I^p, such that f is a homeomorphism in the interior of I^p. A *cellular partition of* X is a partition of X into the disjoint union of cells. A *cell-complex structure on* X, or equivalently, a *cellular decomposition of* X, is a cellular partition such that for each p-dimensional cell $E \subset X$, its boundary ∂E is contained in X^{p-1}, where X^{p-1} is the $p-1$ *skeleton of* X, the union of $p-1$-dimensional cells. All cell complexes considered in the paper will be *finite*, i.e., consist of finitely many cells, and therefore automatically be *CW-complexes*. Cell complexes form a tensor category with respect to cellular maps and direct products. A *cellular operad* is an operad of cell complexes. A *stratification* of a manifold is a decomposition of the manifold into the disjoint union of connected submanifolds, called *strata*, so that the boundary of a stratum is the union of strata of lower dimensions.

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1. The G_∞-operad

1.1. Getzler-Jones' cellular partition

Let us recall the construction of Getzler and Jones [10] of a cellular partition of compactified configuration spaces.

1.1.1. *Fox-Neuwirth cells*

Consider Fox-Neuwirth's cellular decomposition [7] of the one-point compactification of a configuration space $F(\mathbb{C},n) = \mathbb{C}^n \setminus \Delta$, where Δ is the fat diagonal $\cup_{i \neq j}\{x_i = x_j\}$, of $n \geq 1$ distinct points in the complex plane $\mathbb{C}$: The cells, which we will call *Fox-Neuwirth cells*, will be labeled by ordered partitions of the set $\{1,\dots,n\}$ into ordered subsets. For example, $\{3\}\{2,1\}$ denotes a partition of $\{1,2,3\}$ into two subsets: the first subset is $\{3\}$ and the second subset is $\{2,1\}$, ordered so that 2 precedes 1. Partitioning labels $\{1,\dots,n\}$ into ordered subsets reflects grouping points lying on common vertical lines on the plane. Ordering between subsets is the left-to-right order between the vertical lines; ordering within a subset is the bottom-to-top order within the vertical line. For each $n \geq 1$, take the quotient space

$$\mathcal{M}(n) = F(\mathbb{C},n)/\mathbb{R}^2 \rtimes \mathbb{R}^*_+$$

by the action of translations and dilations on $F(\mathbb{C},n)$. The dimension of $\mathcal{M}(n)$ is equal to $2n-3$ for $n \geq 2$ and 0 for $n = 1$. The Fox-Neuwirth cells are obviously invariant under this action, and their quotients, which we will also call *Fox-Neuwirth cells*, make up a cellular decomposition of the one-point compactification $\mathcal{M}(n)^\bullet$ of $\mathcal{M}(n)$. The spaces $\mathcal{M}(n)$ do not form an operad, but one can glue lower $\mathcal{M}(k)$'s to the boundaries of higher $\mathcal{M}(n)$'s to form a topological operad $\underline{\mathcal{M}} = \{\underline{\mathcal{M}}(n) \mid n \geq 2\}$. In fact, the underlying spaces $\underline{\mathcal{M}}(n)$ are smooth manifolds with corners compactifying $\mathcal{M}(n)$, see next section.

All the spaces $D(n)$, $F(\mathbb{C},n)$, $\mathcal{M}(n)$, and $\underline{\mathcal{M}}(n)$ are homotopy equivalent. Moreover D and $\underline{\mathcal{M}}(n)$ are homotopy equivalent operads, see [10].

1.1.2. *Compactified moduli spaces*

The resulting space $\underline{\mathcal{M}}(n)$ is an S^1-bundle over the real compactification $\underline{\mathcal{M}}_{0,n+1}$ of the moduli space $\mathcal{M}_{0,n+1}$ of $n+1$-punctured curves of genus zero, see [10, 12]. The space $\underline{\mathcal{M}}(n)$ can also be interpreted as a "decorated" moduli space, see [12]. Indeed, it can be identified with the moduli space of data $(C; x_1,\dots,x_{n+1}; \tau_1,\dots,\tau_m,\tau_\infty)$, where C is a stable complex complete algebraic curve with $n+1$

punctures $x_1, \dots, x_{n+1}$ and m double points. For each i, $1 \leq i \leq m$, τ_i is the choice of a tangent direction at the ith double point to the irreducible component that is farther away from the "root", i.e., from the component of C containing the puncture $\infty := x_{n+1}$, while τ_∞ is a tangent direction at ∞. The stability of a curve is understood in the sense of Mumford's geometric invariant theory: Each irreducible component of C must be stable, i.e., admit no infinitesimal automorphisms. The operad composition is given by attaching the ∞ puncture on a curve to one of the other punctures on another curve, keeping the tangent direction at each new double point.

1.1.3. *Getzler-Jones cells*

Fox-Neuwirth's cell decomposition of each space $\mathcal{M}(n)^\bullet$ induces a cell partition of the compactification $\underline{\mathcal{M}}(n)$ in the following way. First of all, by an *n-tree* we mean a directed rooted tree with n labeled initial edges, the *leaves*, and one terminal edge, the *root*, each of these $n+1$ edges incident to only one vertex of the tree, such that the number $n(v)$ of the incoming edges for any vertex v is at least two. Define the *tree degree* of an n-tree T as $n - v(T) - 1$, where $v(T)$ is the number of vertices in T. The cells, which we will call *Getzler-Jones cells*, in $\underline{\mathcal{M}}(n)$ are enumerated by pairs (T, p), where T is a tree, labeling a stratum of the boundary of $\underline{\mathcal{M}}(n)$, and p is a function $p(v)$ on the vertices v of the tree T, such that each $p(v)$ is an ordered partition, as in 1.1.1 above, of the set $\operatorname{in}(v)$ of incoming edges for a vertex v of the T. These partitions $p(v)$ label cells in the corresponding open moduli spaces $\mathcal{M}(n(v))$, whose products make up the stratum. However generally speaking, it is not true that this cellular partition is a cell complex: The boundary of a q-dimensional cell does not always lie in the $q-1$-skeleton. We will take the union of certain Getzler-Jones cells to form a stratification of $\underline{\mathcal{M}}(n)$ compatible with the operad structure.

1.2. Stratification of $\underline{\mathcal{M}}$

1.2.1. *Stratification of $\mathcal{M}$*

Consider the following subsets in $\mathcal{M}(n)$.

- For each ordering of the set $\{1, \dots, n\}$, consider the corresponding Fox-Neuwirth cell. It consists of configurations of n points on a vertical line in the prescribed order going from bottom to top.

- For each ordered partition of the set $\{1, \dots, n\}$ into two parts, consider the corresponding Fox-Neuwirth cell. It consists of configurations of n points on two vertical lines in the prescribed order going from bottom to top and from left to right.

- The complement to the union of subsets of the above two types.

These subsets obviously form a stratification of the manifold $\mathcal{M}(n)$. We will also need the following filtration of $\mathcal{M}(n)$ into three closed subsets: The closure J_1 of the union of the strata of the first type, the closure J_2 of the union of the strata of the second type, and J_3 which is all of $\mathscr{M}(n)$.

Adding the point at ∞ to each filtration component and setting $J_0 = \{\infty\}$, we get a filtration of the pointed space $\mathscr{M}(n)^\bullet$ for $n \geq 3$:

$$J_0 \subset J_1 \subset J_2 \subset J_3 = \mathscr{M}(n)^\bullet. \tag{1}$$

For $n = 2$, the space $\mathscr{M}(2)$ is diffeomorphic to S^1 and thereby compact. In this case we will not add a point ∞ to any filtration components and will set $J_0 = \emptyset$.

Consider the corresponding homological spectral sequence converging to $H_\bullet(\mathscr{M}(n)^\bullet, \infty)$, which is naturally isomorphic to $H^\bullet(\mathscr{M}(n))$ by Poincaré-Lefschetz duality, all coefficients being taken in k. The term E^1 may be identified with the sum of $E^1_{p,q} = H_{p+q}(J_p, J_{p-1})$, $1 \leq p \leq 3$, $-p \leq q \leq 2n-p-3$.

Proposition 1.1 *The homological spectral sequence associated to the filtration* (1) *collapses at* E^2.

Proof. Since $J_2 \setminus J_1$ and $J_1 \setminus J_0$ are disjoint unions of $n!(n-1)$ and $n!$ cells of dimension $n-1$ and $n-2$, respectively,

$$\dim H_{2+q}(J_2, J_1) = \begin{cases} n!(n-1) & \text{for } 2+q = n-1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\dim H_{1+q}(J_1, J_0) = \begin{cases} n! & \text{for } 1+q = n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Note also that with respect to the Fox-Neuwirth cellular decomposition of J_3, the subspace J_2 is a cell subcomplex. The dimensions of cells in the complement of J_2 run from n through $2n-3$. Thus, $H_{p+q}(J_3, J_2) = 0$, unless $n \leq p+q \leq 2n-3$. Observe that for any pair (p,q) and integer $r \geq 2$, either $E^r_{p,q}$ or $E^r_{p-r,q+r-1}$ is 0. Thus the spectral-sequence differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ will all vanish for $r \geq 2$, which implies the collapse of the spectral sequence at E^2. ■

1.2.2. *Stratification of $\underline{\mathscr{M}}$*

First of all, stratify $\underline{\mathscr{M}}(n)$ by the topological type of the stable algebraic curve. We will refer to this stratification as *coarse*. Each stratum S_T will correspond to a tree T and be isomorphic to the product of the spaces $\mathscr{M}(n(v))$ over the vertices v of the tree as in 1.1.3. Then subdivide the coarse stratification as follows. For each space $\mathscr{M}(n(v))$, take the stratification into three types of strata as in the previous section. The products of these strata over the set of vertices v of T will form a finer partition of the space $\underline{\mathscr{M}}(n)$. Figure 2 pictures the part obtained as the product of two strata $J_2 \setminus J_1$ in $\mathscr{M}(3)$ and $J_1 \setminus J_0$ in $\mathscr{M}(2)$.

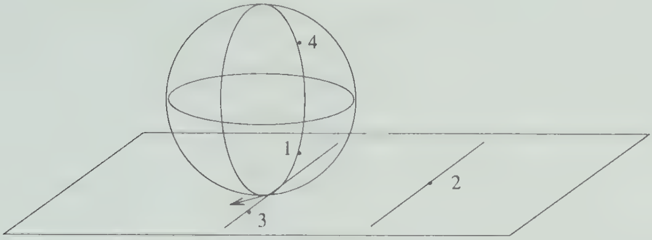

Figure 2. A stratum: a 3D picture.

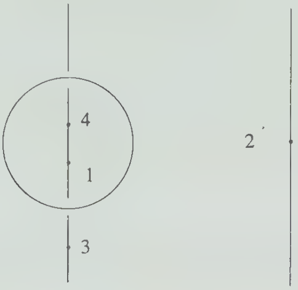

Figure 3. The same stratum: a 2D picture.

Here the plane denotes the irreducible component of the stable curve with the puncture ∞ placed at ∞. The other irreducible component is the sphere. The arrow at the double point determines the positive direction of the real axis on the sphere. We prefer to work with the following replacement of the above figure, which one may think of as projection of the three-dimensional figure onto the plane after rotating the sphere with the arrow, so that the arrow points in the direction of the positive real axis on the plane. The circle may be thought of as a magnifying glass through which the observer living on the plane sees what happens in the infinitesimal world at the double point, see Figure 3.

We will usually think of either of these figures as not a single configuration of points, but rather the set of all configurations of points which are in the relative position determined by the figure.

The above partition of the space $\underline{\mathcal{M}}(n)$ will in fact be a stratification. The reason is that the boundary of each part may be split into the union of the boundary within the coarse stratum S_T and the boundary within the boundary of S_T. The boundary of the first type is the union of other strata within the coarse stratum S_T, because we used a stratification of each $\mathcal{M}(n(v))$ to construct our partition of $\underline{\mathcal{M}}(n)$. The boundary of the second type is obtained by letting a few groups of points on the irreducible components of the stable curve bubble off, forming new components attached at double points. If all the points on a component lie on a single vertical line, the points on the limiting components will also group on a single vertical

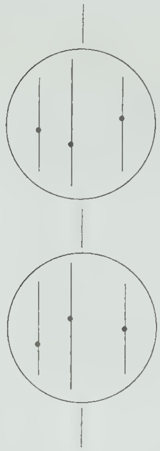

Figure 4. A proper subset of this cell lies in the boundary of another cell.

line within each bubble, and again, the whole fine part will be in the boundary. If all the points on a component lie on two vertical lines, then the points on the components that bubble off will lie on two or one vertical lines, and again, the whole fine part will be in the boundary, because by translations and dilations, one can always match up the vertical lines on different components (if less than three on each), for example, pass through points 0 or 1 on the real axis. Since $J_3 \setminus J_2$ comprises the top strata in $\mathscr{M}(n(v))$, the closure of this component of the stratum will be all $\underline{\mathscr{M}}(n(v))$, therefore, the boundary is $J_2 \cup \partial \underline{\mathscr{M}}(n(v))$, which is the union of lower-dimensional strata.

Remark 3 Note that if we extended our partition to a cell partition by taking all the Getzler-Jones cells, the boundary of a cell would not be the union of other cells, in general. For example, take the Fox-Neuwirth cell formed by six points on two vertical lines, three points on each. Its boundary has a nonempty intersection with the Getzler-Jones cell sketched on Figure 4.

However, only part of the this Getzler-Jones cell will be in the boundary of the Fox-Neuwirth cell. This part will consist of those positions of three vertical lines on the two bubbles that can be brought together by the action of $\mathbb{R}^2 \rtimes \mathbb{R}^*_+$ on a single copy of the Riemann sphere. This is exactly the problem with Getzler-Jones' partition not being a cell complex. Our stratification is designed to get around this problem.

Now form two filtrations $\dots \subset F^p \subset F^{p+1} \subset \dots$ and $\dots \subset \overline{F}^p \subset \overline{F}^{p+1} \subset \dots$ by taking F^p (respectively, $\overline{F}^p$) to be the closure of the union of coarse (respectively, fine) strata of dimension p. Consider the corresponding spectral sequences, which converge to $H_\bullet(\underline{\mathscr{M}}(n))$ and whose first terms are

$$E^1_{p,q} = H_{p+q}(F^p, F^{p-1}),$$
$$\overline{E}^1_{p,q} = H_{p+q}(\overline{F}^p, \overline{F}^{p-1}).$$

1.2.3. *Operad properties of the first spectral sequence* E^r
Note that the operad composition respects the two filtrations of spaces $\underline{\mathscr{M}}(n)$, $n \geq 1$. Therefore, we are getting two operads of spectral sequences, cf. [20]. In particular, the first terms E^1 and $\overline{E}^1$ of the spectral sequences form operads of complexes. There is a purely algebraic interpretation of the first operad E^1, noticed by Getzler and Jones in [10].

Proposition 1.2. (Getzler and Jones) *The operad E^1 is naturally isomorphic to the cobar construction of the Gerstenhaber operad $e_2 = H_\bullet(D)$.*

Proof. Notice that for the space $\underline{\mathscr{M}}(n)$, $n \geq 2$, the group $E^1_{p,q} = H_{p+q}(F^p, F^{p-1})$ is naturally isomorphic to $H^{-q}(F^p \setminus F^{p-1})$ by Poincaré-Lefschetz duality. The space $F^p \setminus F^{p-1}$ is the disjoint union of the strata S_T, T running over the set of n-trees T which have exactly $2n-2-p$ vertices. Each stratum S_T is naturally isomorphic to the product of spaces $\mathscr{M}(n(v))$ over the set of vertices v of the tree T. Each space $\mathscr{M}(n(v))$ is homotopy equivalent to the configuration space $D(n(v))$ of $n(v)$ little disks. Thus

$$E^1_{p,q} = \bigoplus_{n\text{-trees } T} \left(\bigotimes_{v \in T} H^\bullet(D(n(v))) \right)^{-q}, \tag{2}$$

where the superscript $-q$ means the component of degree $-q$. The differential

$$d^1 : E^1_{p,q} \to E^1_{p-1,q}$$

takes the component $\bigotimes_{v\in T} H^\bullet(D(n(v)))$ corresponding to a n-tree T to the sum of components $\bigotimes_{v\in\widehat{T}} H^\bullet(D(n(v)))$ over all n-trees $\widehat{T}$ such that the tree T may be obtained by contracting an interior edge of $\widehat{T}$, merging two adjacent vertices v_1 and v_2 on the tree $\widehat{T}$ into a single vertex v_3 of T. The matrix element $d^1_{T\widehat{T}}$ of the differential is induced (up to a sign, which is treated below) by the map

$$\circ_i^* : H^\bullet(D(n(v_3))) \to H^\bullet(D(n(v_1))) \otimes H^\bullet(D(n(v_2)))$$

which is the dual of the corresponding operad structure map

$$\circ_i : H_\bullet(D(n(v_1))) \otimes H_\bullet(D(n(v_2))) \to H_\bullet(D(n(v_3))).$$

This map is induced on homology by the map

$$\circ_i : D(n(v_1)) \times D(n(v_2)) \to D(n(v_3))$$

gluing the unit disk with a configuration of $n(v_2)$ little disks into the ith little disk in a configuration of $n(v_2)$ little disks, i corresponding to the contracted edge in $\mathrm{in}(v_1)$, if we assume that the contracted edge is directed from v_2 to v_1.

The sign for $d^1_{T\widehat{T}}$ comes from the choice of orientation of the strata S_T. This orientation may be chosen by ordering all the edges of each n-tree T, except the root edge. The orientation on the stratum S_T is then given by ordering the x and the y coordinates of the points in $\mathbb{C}$ corresponding to the edges of T, according to the order of the edges, skipping for each vertex v the coordinates of the point corresponding to the first edge and the x coordinate of the point corresponding to the second edge — remember that $\mathscr{M}(n) = F(\mathbb{C},n)/\mathbb{R}^2 \rtimes \mathbb{R}^*_+$. Then the compatibility of orientations on S_T and $S_{\widehat{T}}$ implies that $d^1_{T\widehat{T}}$ is $\circ_i^*$ multiplied by the sign of the permutation from the ordered set of edges of $\widehat{T}$ to the ordered set $\{e, \text{edges of } T\}$, where e is the contracted edge in $\widehat{T}$.

This description of E^1 in terms of trees and the operad $H_\bullet(D)$ of vector spaces means that E^1 is the cobar construction of the operad $H_\bullet(D)$, just by the definition of Ginzburg-Kapranov [11]. ■

Corollary 1.3. (Getzler and Jones, Markl) *The operad E^1 is a free resolution of the G-operad e_2, i.e., there is a morphism of operads $E^1 \to e_2$, inducing an isomorphism on homology and E^1 is free as an operad of graded vector spaces. Moreover E^1 is a minimal model of e_2.*

Remark 4 Here we sketch a proof due to Markl [22], which is different from that of Getzler and Jones [10]. The fact that E^1 is a minimal model of e_2 was first noticed by Markl [22].

Proof. Using the description of the operad e_2 in terms of generators and relations, it is a straightforward exercise to check that the quadratic dual of e_2 is again e_2 up to a shift of grading and the change of it to the opposite. Then from the proof of Proposition 1.2, one can identify E^1 with the cobar construction of $e_2^!$. The natural homomorphism, see Lemma 4.1.2 of [11], which easily generalizes to the graded case, from the cobar construction of an operad $\mathscr{P}$ to the quadratic dual $\mathscr{P}^!$ gives for $\mathscr{P} = e_2^!$ a morphism of operads

$$E^1 \to e_2. \tag{3}$$

It is also known that the operad e_2 is Koszul, see [10] or a purely algebraic proof by Markl [22]. This means (by definition of [11]) that the morphism (3) is a quasi-isomorphism.

According to Markl [23], the cobar construction of the quadratic dual of a Koszul operad (see [11]) is a minimal model of that operad. Applied to e_2, this implies that E^1 is a minimal model of e_2. ■

Definition 1.1 *The G_∞-*operad *is the operad E^1. A G_∞-*algebra *is an algebra over the G_∞-operad.*

1.2.4. *Operad properties of the second spectral sequence $\overline{E}^r$*

Theorem 1.4 *The operad $\overline{E}^1$ is free as an operad of graded vector spaces, and its homology is isomorphic to e_2.*

Proof. 1. First of all, let us prove that $\overline{E}^1$ is free. Recall that

$$\overline{E}^1_{p,q} = H_{p+q}(\overline{F}^p, \overline{F}^{p-1}) = H^{-q}(\overline{F}^p \setminus \overline{F}^{p-1}).$$

Since $\overline{F}^p$'s make up a stratification of $\underline{\mathscr{M}}(n)$ (see Section 1.2.2), we have

$$\coprod_p \overline{F}^p \setminus \overline{F}^{p-1} = \coprod_{n\text{-trees } T} \prod_{v \in T} \coprod_{i=1}^{3} J_i(n(v)) \setminus J_{i-1}(n(v)),$$

where for each $l \geq 2$,

$$J_0(l) \subset J_1(l) \subset J_2(l) \subset J_3(l) = \mathscr{M}(l)^\bullet \tag{4}$$

is the filtration from Section 1.2.1. Therefore, passing to cohomology, we have

$$\overline{E}^1_{\bullet,\bullet} = \bigoplus_{n\text{-trees } T} \bigotimes_{v \in T} \bigoplus_{i=1}^{3} H^\bullet(J_i(n(v)) \setminus J_{i-1}(n(v))),$$

which by definition means that $\overline{E}^1_{\bullet,\bullet}$ is a free operad generated by the collection

$$\bigoplus_{i=1}^{3} H^\bullet(J_i(n) \setminus J_{i-1}(n)) = \bigoplus_{i=1}^{3} H_\bullet(J_i(n), J_{i-1}(n)) \tag{5}$$

of graded vector spaces with an action of the symmetric group S_n, $n \geq 2$.

2. The next step is to show that the spectral sequence $\overline{E}^r$ collapses at the second term $\overline{E}^2$. In order to show that the cohomology of $\overline{E}^1$ is $\overline{E}^\infty = e_2$, regard $\overline{E}^1$ as a filtered complex, with the kth filtration component defined by the tree degree $n - v(T) - 1 \leq k$. We will compute the homology of $\overline{E}^1$ using the spectral sequence associated with this filtration. The first term of this spectral sequence is $\bigoplus_{n\text{-trees } T} \bigotimes_{v \in T} H^\bullet(\mathscr{M}(n(v)))$ with the Gysin homomorphism as the differential, because of Proposition 1.1, that is the first term E^1 of the spectral sequence associated with the filtration $F^\bullet$, see (2). Corollary 1.3 shows that the homology of E^1 is isomorphic to e_2. By construction this is the second term of the spectral sequence associated to the filtered complex $\overline{E}^1$ and the spectral sequence converges to the homology $\overline{E}^2$ of d^1. On the other hand, e_2 is the ∞ term $\overline{E}^\infty$. Thus e_2 is the second term of a spectral sequence converging to the second term of another spectral

sequence converging to the same e_2. This implies that both spectral sequences collapse at the second terms. Therefore, the homology of $\overline{E}^1$ is e_2. ■

2. The B_∞-operad

2.1. B_∞-algebras and the B_∞-operad

Let $V = \bigoplus_{n\in\mathbb{Z}} V^n$ be a graded vector space over a field k of characteristic zero, $V[1]$ its *desuspension*: $V[1] = \bigoplus_{n\in\mathbb{Z}} V[1]^n$, where $V[1]^n = V^{n+1}$, and $TV[1] = \sum_{p=0}^{\infty}(V[1])^{\otimes p}$ the *tensor coalgebra* on $V[1]$. We will adopt the standard notation $[a_1|\dots|a_p]$ for an element $a_1 \otimes \cdots \otimes a_p \in (V[1])^{\otimes p} \subset TV[1]$. By definition $|[a_1|\dots|a_p]| = |a_1| + \cdots + |a_p| - p$, where $|a|$ denotes the degree of an element a in a graded vector space. The graded coalgebra structure on $TV[1]$ is given by the coproduct $\Delta : TV[1] \to TV[1] \otimes TV[1]$,

$$\Delta[a_1|\dots|a_p] = \sum_{i=0}^{p} [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_p],$$

for which the natural augmentation $TV[1] \to k$ is a counit.

We will be interested in studying a certain DG bialgebra structure on $TV[1]$. Here a *DG bialgebra* is an algebra A with a unit, the structure of a coalgebra, and a differential $D : A \to A[1]$, $D^2 = 0$, such that D is a graded derivation and coderivation and the comultiplication $A \to A \otimes A$ is a morphism of algebras.

Definition 2.1. (H. J. Baues [3]) *A* B_∞-algebra *structure on a graded vector space V is the structure of a DG bialgebra on the tensor coalgebra $TV[1]$, such that the element $[] \in (V[1])^{\otimes 0} \subset TV[1]$ is a unit element.*

Since the tensor coalgebra is cofree and both the differential $D : TV[1] \to TV[1]$ and the product $M : TV[1] \otimes TV[1] \to TV[1]$ are respect the coproduct, they are determined by the compositions

$$\operatorname{pr} D = \sum_{k=0}^{\infty} M_k : TV[1] \to V[2]$$

and

$$\operatorname{pr} M = \sum_{k,l=0}^{\infty} M_{k,l} : TV[1] \otimes TV[1] \to V[1]$$

with the natural projection $\operatorname{pr} : TV[1] \to V[1]$. The condition $D^2 = 0$ can be rewritten as a collection of identities for the operations M_k, the associativity condition for M and the unit axiom for $[]$ as a collection of identities for the operations $M_{k,l}$ and the derivation property for D with respect to M as a collection of identities between $M_{k,l}$ and M_k. The restriction M_0 of D to $V[1]^{\otimes 0}$ must vanish, because D

must annihilate the unit $[]$. The equation $D^2 = 0$ then implies $M_1^2 = 0$, which means $d = M_1$ must be a differential on the graded vector space V, defining the structure of a complex with a differential of degree 1. Thus, a B_∞-algebra structure on a graded vector space is equivalent to a differential d and a collection of multilinear operations M_k of degree $|M_k| = 2-k$ and $M_{k,l}$ of degree $|M_{k,l}| = 1-k-l$ satisfying certain identities, i.e., the structure of an algebra over the DG operad generated by M_k, $k \geq 2$, and $M_{k,l}$, $k,l \geq 0$, with those identities being the defining relations. We will call this operad the *B_∞-operad.*

2.2. The algebraic description of the B_∞-operad

Here we will describe the B_∞-operad explicitly. We will use this description in the next section to show that the B_∞-operad is a quotient of the operad $\overline{E}^1$, associated to the little disks operad. Just to make the formulas more transparent, we will describe the identities satisfied by the operations M_k, $k \geq 0$, and $M_{k,l}$, $k,l \geq 0$, in a B_∞-algebra V. As we already noticed, $M_0 = 0$ and M_1 is a differential d on V. We will adopt the following convention:

$$M_k(a_1,\ldots,a_k) := (-1)^{(k-1)|a_1|+(k-2)|a_2|+\cdots+|a_{k-1}|} M_k[a_1|\ldots|a_k],$$

which morally means that the vertical bar $|$ has degree one and on the left-hand side all the bars are moved between M_k and a_1. Here $|a_i|$ denotes the degree of a_i in V. However, we set

$$M_{k,l}(a_1,\ldots,a_k;b_1,\ldots,b_l) := M_{k,l}([a_1|\ldots|a_k]\otimes[b_1|\ldots|b_l]).$$

2.2.1. $D^2 = 0$

The condition $D^2 = 0$ is equivalent for the operations M_k, $k \geq 1$, to define an A_∞-structure:

$$\sum_{i+j=n+1}\sum_{k=0}^{n-j}(-1)^{\varepsilon} M_i(a_1,\ldots,a_k, M_j(a_{k+1},\ldots,a_{k+j}),\ldots,a_n) = 0, \quad n \geq 1, \tag{6}$$

where $\varepsilon = (i+1)j + (j+1)k + i|a_1| + (i-1)|a_2| + \cdots + (i-k+1)|a_k| + (n-k-1)|a_{k+1}| + (n-k-2)|a_{k+2}| + \cdots + |a_{n-1}|$ and $a_1,\ldots,a_n \in V$. In fact, the sign ε is obtained as $|a_1| + \cdots + |a_k| - k$ plus the sign coming from moving the vertical bars in any occurrence of $M_p[a_1|\ldots|a_p]$ to the place between M_p and a_1, thinking of a bar as having degree 1.

2.2.2. $[]$ *is a unit for* M

This is equivalent to $M_{1,0} = M_{0,1} = \mathrm{id}$, $M_{k,0} = M_{0,k} = 0$ for $k \neq 1$.

2.2.3. *The associativity of M*

The associativity of $M = \sum_{k,l\geq 0} M_{k,l}$ is equivalent to the following identities

$$\begin{aligned}
&\sum_{r=1}^{l+m} \sum_{\substack{l_1+\cdots+l_r=l\\ m_1+\cdots+m_r=m}} (-1)^{\varepsilon} M_{k,r}(a_1,\dots,a_k; M_{l_1,m_1}(b_1,\dots,b_{l_1};c_1,\dots,c_{m_1}),\\
&\qquad \dots, M_{l_r,m_r}(b_{l_1+\cdots+l_{r-1}+1},\dots,b_l;c_{m_1+\cdots+m_{r-1}+1},\dots,c_m))\\
&\qquad = \sum_{s=1}^{k+l} \sum_{\substack{k_1+\cdots+k_s=k\\ l_1+\cdots+l_s=l}} (-1)^{\delta} M_{s,m}(M_{k_1,l_1}(a_1,\dots,a_{k_1};b_1,\dots,b_{l_1}),\dots,\\
&\qquad M_{k_s,l_s}(a_{k_1+\cdots+k_{s-1}+1},\dots,a_k;b_{l_1+\cdots+l_{s-1}+1},\dots,b_l);c_1,\dots,c_m)
\end{aligned} \tag{7}$$

for $a_1,\dots,a_k, b_1,\dots,b_l$, and $c_1,\dots,c_m$ in V. The sign $(-1)^{\varepsilon}$ is the sign picked up by reordering $[a_1|\dots|a_k|b_1|\dots|b_l|c_1|\dots|c_m]$ into $[a_1|\dots|a_k|b_1|\dots\ |b_{l_1}|c_1|\dots|c_{m_1}|\dots |b_{l_1+\cdots+l_{r-1}+1}|\dots|b_l|c_{m_1+\cdots+m_{r-1}+1}|\dots|c_m]$ in the graded vector space $TV[1]$. Similarly, $(-1)^{\delta}$ is the sign of reordering $[a_1|\dots|a_k|b_1|\dots|b_l|c_1|\dots|c_m]$ into $[a_1|\dots|a_{k_1}|b_1|\dots|b_{l_1}|\dots|a_{k_1+\cdots+k_{s-1}+1}|\dots|a_k|b_{l_1+\cdots+l_{s-1}+1}|\dots|b_l|c_1|\dots|c_m]$.

2.2.4. *The Leibniz rule for D with respect to M*

The fact that D is a derivation of the product M on $TV[1]$ is equivalent to the following identities

$$\begin{aligned}
&\sum_{n=1}^{k+l} \sum_{\substack{k_1+\cdots+k_n=k\\ l_1+\cdots+l_n=l}} (-1)^{\varepsilon} M_n(M_{k_1,l_1}(a_1,\dots,a_{k_1};b_1,\dots,b_{l_1}),\dots,\\
&\qquad\qquad M_{k_n,l_n}(a_{k_1+\cdots+k_{n-1}+1},\dots,a_k;b_{l_1+\cdots+l_{n-1}+1},\dots,b_l))\\
&= \sum_{r=1}^{k} \sum_{i=0}^{k-r} (-1)^{\delta} M_{k-r+1,l}(a_1,\dots,a_i,M_r(a_{i+1},\dots,a_{i+r}),\dots,a_k;b_1,\dots,b_l)\\
&\qquad + (-1)^{|a_1|+\dots|a_k|-k} \sum_{s=1}^{l} \sum_{i=0}^{l-s} (-1)^{\eta} M_{k,l-s+1}(a_1,\dots,a_k;b_1,\dots,b_i,\\
&\qquad\qquad M_s(b_{i+1},\dots,b_{i+s}),\dots,b_l)
\end{aligned} \tag{8}$$

for $a_1,\dots,a_k$ and $b_1,\dots,b_l$ in V. The sign $(-1)^{\varepsilon}$ is the sign of reordering $[a_1|\dots|a_k| b_1|\dots|b_l]$ into $[a_1|\dots|a_{k_1}|b_1|\dots|b_{l_1}|\dots|a_{k_1+\cdots+k_{n-1}+1}|\dots|a_k|b_{l_1+\cdots+l_{n-1}+1}|\dots|b_l]$ in $TV[1]$ multiplied by the sign of moving $n-1$ bars between $M_{k_1,l_1}(\dots),\dots, M_{k_n,l_n}(\dots)$ to the place between M_n and M_{k_1,l_1}. The sign $(-1)^{\delta}$ is equal to $|a_1| + \dots + |a_i| - i$ plus the sign coming from moving the vertical bars in $M_r[a_{i+1}|\dots|a_{i+r}]$ to the place between M_r and a_{i+1}. Similarly, the sign $(-1)^{\eta}$ is equal to $|b_1| + \dots|b_i| - i$ plus the sign coming from moving the vertical bars in $M_s[b_{i+1}|\dots|b_{i+s}]$ to the place between M_s and b_{i+1}.

Remark 5 A few "lower" identities including the derivation property of the differential $d = M_1$ with respect to the "dot" product M_2 and identities providing homotopies for such classical identities for binary operations as the commutativity and the associativity of the "dot" product, the homotopy left and right Leibniz rules for the "circle" product $M_{1,1}$ with respect to the "dot" product, and the homotopy Jacobi identity for the "bracket" $[a,b] = M_{1,1}(a,b) - (-1)^{(|a|-1)(|b|-1)}M_{1,1}(b,a)$ were written out explicitly in [21, Section 4.2]. Strictly speaking, those identities were claimed to be identities for another type of algebra, which later turned out to be nonextant. However, one can see from Theorem 2.1 of the next section, that those identities are satisfied in a B_∞-algebra.

2.3. Relation between the B_∞-operad and $\overline{E}^1$

The algebraic description above of the B_∞-operad might be a good exercise in tensor algebra, but is far from inspiring. However, everything falls into its place, when geometry comes into play. Since $\overline{E}^1$ is a homological operad (the degree of the differential is -1), if B_∞ is a cohomological operad (the degree of the differential is $+1$), let us change the grading on $\overline{E}^1$ to the opposite one, an element of degree k will be assigned degree $-k$, from now on, so that the differential on $\overline{E}^1$ is of degree $+1$.

Theorem 2.1 *There exists a surjective morphism of DG operads $\overline{E}^1 \to B_\infty$.*

Proof. The operad $\overline{E}^1$ is freely generated by the spaces $\bigoplus_{i=1}^3 H_\bullet(J_i(n), J_{i-1}(n))$, $n \geq 2$, see (5). Thus to define a morphism $\overline{E}^1 \to B_\infty$, it suffices to define maps

$$\bigoplus_{i=1}^{3} H_\bullet(J_i(n), J_{i-1}(n)) \to B_\infty(n), \qquad n \geq 2, \tag{9}$$

respecting the gradings, the symmetric group actions, and the differentials.

The complements $J_1 \setminus J_0$ and $J_2 \setminus J_1$ are disjoint unions of Fox-Neuwirth cells, and the spaces $H_\bullet(J_1, J_0)$ and $H_\bullet(J_2, J_1)$ have the Fox-Neuwirth cells of the types $\{i_1, \ldots, i_n\}$ and $\{i_1, \ldots, i_p\}\{i_{p+1}, \ldots, i_n\}$, respectively, see 1.1.1, as natural bases. Define the maps (9) as follows:

$$\{1,2,\ldots,n\} \mapsto M_n \qquad \text{for } n \geq 2, \tag{10}$$

$$\{1,2,\ldots,k\}\{k+1,\ldots,k+l\} \mapsto M_{k,l} \qquad \text{for } k,l \geq 1, \tag{11}$$

permutations of the cells mapping to permutations of the generators M_n and $M_{k,l}$ of the B_∞-operad. Finally define

$$H_\bullet(J_3(n), J_2(n)) \to B_\infty(n), \qquad n \geq 2,$$

as zero.

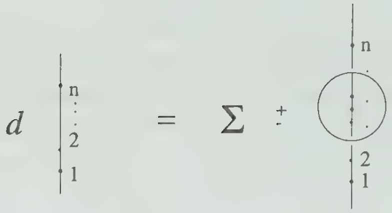

Figure 5. The boundary of a cell in J_1.

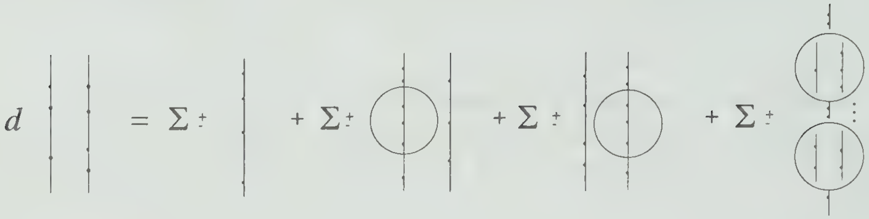

Figure 6. The boundary of a cell in $J_2 \setminus J_1$.

Since $\dim\{1,\dots,n\} = n-2 = -|M_n|$ and $\dim\{1,\dots,k\}\{1,\dots,l\} = k+l-1 = -|M_{k,l}|$ and the action of the symmetric groups is respected by construction, the maps (10) and (11) define a morphism $\overline{E}^1 \to B_\infty$ of graded operads. The only thing which remains to be checked is the compatibility of this morphism with the differentials.

We will compute the boundary (differential) $d := d^1$ on $\overline{E}^1$. Each space $J_i \setminus J_{i-1}$, $i = 1,2,3$, is a disjoint union of Fox-Neuwirth cells, which form a basis of the relative homology $H_\bullet(J_i, J_{i-1})$. We will study the action of d on this basis.

$i = 1$. Let us start with $i = 1$, when the points in a Fox-Neuwirth cell group on a single vertical line. For $n \geq 2$ the boundary of the cell $\{1,2,\dots,n\}$ in $\overline{E}^1$ may be computed as shown on Figure 5.

where the left-hand side denotes the boundary of the cell $\{1,\dots,n\}$ and the right-hand side denotes a linear combination of Getzler-Jones cells obtained as operad compositions $\{1,\dots,k\} \circ_i \{1,\dots,l\}$, $k,l \geq 2$, of two Fox-Neuwirth cells. This equation turns into equation (6), where all the terms with $i > 1$ or $j > 1$ are moved to the right-hand side. The signs here and henceforth in the proof are compatible with the signs in (6)–(8), if the orientations on Getzler-Jones cells are chosen as in the proof of Proposition 1.2.

$i = 2$. Cells for $i = 2$ are configurations of points on two vertical lines. The boundary of a cell $\{1,\dots k\}\{k+1,\dots,k+l\}$ may be described as shown on Figure 6.

This equation translates under the correspondence (10), (11) into the identity (8) in which all terms but those containing $d = M_1$ are moved to the right-hand side.

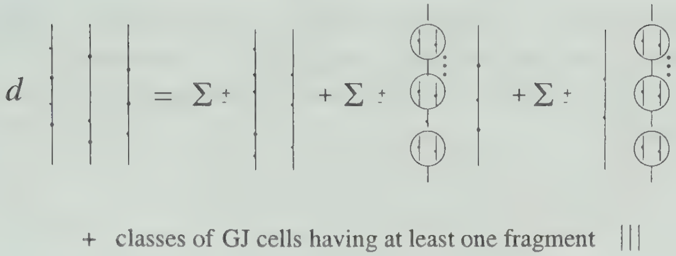

Figure 7. The boundary of a cell in $J_3 \setminus J_2$.

Note that the first sum on the figure corresponds to the terms $M_{k+l}(c_1,\dots,c_{k+l})$, where $c_1,\dots,c_{k+l}$ is a shuffle of $\{a_1,\dots,a_k\}$ and $\{b_1,\dots,b_l\}$, which show up on the left-hand side of (8) for all pairs (k_i,l_i) being $(0,1)$ or $(1,0)$. The rest of the left-hand side of (8) is the last term on the figure above.

$i = 3$. Cells for $i = 3$ are configurations of points on at least three different vertical lines. The boundary of a cell with at least four vertical lines will produce the identity $0 = 0$ under the morphism $\overline{E}^1 \to B_\infty$, because of a dimension argument: The boundary of such cell has a dimension $\geq n$, while (multiple) operad compositions of Fox-Neuwirth cells from J_2 will have dimensions $\leq n-1$. Therefore, the boundary of a cell with at least four vertical lines will have no terms which are compositions of Fox-Neuwirth cells from J_2. Thus, the only nontrivial identity to be checked in B_∞ comes from the lowest Fox-Neuwirth cells in $J_3 \setminus J_2$, those made out of configurations of points on three vertical lines.

Figure 7 describes the differential of such cell in $\overline{E}^1$.

Under the morphism $\overline{E}^1 \to B_\infty$, this identity turns into (7) where all terms are moved to the right-hand side. Note that the first sum on the figure corresponds to the terms in (7) where all (l_i,m_i) or all (k_i,l_i) are either $(0,1)$ or $(1,0)$. ■

Corollary 2.2 *The operad $\overline{E}^1$ is formal, i.e., quasi-isomorphic to its homology e_2. There is a morphism of operads $E^1 \to \overline{E}^1$, unique up to homotopy.*

Proof. In [26] Tamarkin has constructed an operad morphism $B_\infty \to e_2$, which is surjective on homology. Composing it with our morphism $\overline{E}^1 \to B_\infty$, we get a morphism $\phi : \overline{E}^1 \to e_2$. To prove the formality of $\overline{E}^1$, it is enough to show that it is a quasi-isomorphism. Moreover, it suffices to show that ϕ is surjective on homology, because of a graded dimension argument.

Note that the operad e_2 is generated by $e_2(2)$, therefore we just need to show that the second component $\phi(2) : \overline{E}^1(2) \to e_2(2)$ of ϕ is surjective on homology. The morphism ϕ is a composition $\overline{E}^1 \to B_\infty \to e_2$. According to Tamarkin [26, Theorem 4.2.1], the induced homology morphism $H_\bullet(B_\infty(2)) \to e_2(2)$ is an isomorphism. On the other hand, notice that our morphism $\overline{E}^1(2) \to B_\infty(2)$ is an isomorphism, because $J_3(2) = J_2(2)$. In particular, it induces an isomorphism $H_\bullet(\overline{E}^1(2)) \to H_\bullet(B_\infty(2))$ of the homology. Thus, the composition $H_\bullet(\phi) : H_\bullet(\overline{E}^1(2)) \to e_2(2)$ is an isomorphism, which shows that $\overline{E}^1$ is formal.

The existence of a unique up to homotopy morphism $E^1 \to \overline{E}^1$ follows from the fact that both operads are quasi-isomorphic to e_2 and E^1 is a minimal model of e_2, see Corollary 1.3. ■

Question 2.3 *Is the homology of the B_∞-operad isomorphic to the G-operad e_2? If yes, it will automatically be formal via Tamarkin's morphism $B_\infty \to e_2$.*

3. Action on the Hochschild complex

3.1. The Hochschild complex of an associative algebra

Let us recall some notions related to the Hochschild complex and some properties of it. Let A be an associative algebra and $C^n(A,A) = \operatorname{Hom}(A^{\otimes n}, A)$ its *Hochschild complex* with the *Hochschild differential*:

$$\begin{aligned}(dx)(a_1,\dots,a_{n+1}) := {} & a_1 x(a_2,\dots,a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i x(a_1,\dots,a_{i-1},a_i a_{i+1},a_{i+2},\dots,a_{n+1}) \\ & - (-1)^n x(a_1,\dots,a_n)a_{n+1},\end{aligned} \tag{12}$$

for $x \in C^n(A,A)$, $a_1,\dots,a_{n+1} \in A$. The sign $(-1)^n$ above is equal to $(-1)^{|x|}$, $|x| := n$ being the *degree* of the cochain x in the Hochschild complex $C^\bullet = C^\bullet(A,A)$.

We will use the following operations on the Hochschild complex. The *dot product* is defined as the usual cup product:

$$(x \cdot y)(a_1,\dots,a_{k+l}) = x(a_1,\dots,a_k)y(a_{k+1},\dots,a_{k+l}) \tag{13}$$

for any k- and l-cochains x and y and $a_i \in A$. The following collection of multilinear operations, called *braces*, cf. Kadeishvili [14] and Getzler [9], on $C^\bullet$ is defined as

$$\begin{aligned}&\{x\}\{x_1,\dots,x_n\}(a_1,\dots,a_m) := \\ &\quad \sum (-1)^\varepsilon x(a_1,\dots,a_{i_1},x_1(a_{i_1+1},\dots),\dots,a_{i_n},x_n(a_{i_n+1},\dots),\dots,a_m),\end{aligned}$$

for $x, x_1,\dots,x_n \in C^\bullet$, $a_1,\dots,a_m \in A$, where the summation runs over all possible substitutions of $x_1,\dots,x_n$ into x in the prescribed order and $\varepsilon := \sum_{p=1}^{n}(|x_p|-1)i_p$. The braces $\{x\}\{x_1,\dots,x_n\}$ are homogeneous of degree $-n$, i.e., $|\{x\}\{x_1,\dots,x_n\}| = |x| + |x_1| + \dots + |x_n| - n$. We will also adopt the following notation:

$$x \circ y := \{x\}\{y\}, \qquad [x,y] := x \circ y - (-1)^{(|x|-1)(|y|-1)} y \circ x.$$

The *G-bracket* $[x,y]$ defines the structure of a G-algebra on the Hochschild cohomology $H^\bullet(A,A)$. The bracket was introduced by Gerstenhaber [8] in order to describe the obstruction for extending a first-order deformation of the algebra A to the second order. The following definition of the bracket is due to Stasheff [25]. Considering the tensor coalgebra $T(A) = \bigoplus_{n=0}^{\infty} A^{\otimes n}$ with the comultiplication $\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n)$, we can identify the Hochschild cochains $\mathrm{Hom}(A^{\otimes n}, A)$ with the graded coderivations $\mathrm{Coder}\, T(A)$ of the tensor coalgebra $T(A)$. Then the G-bracket $[x,y]$ is defined as the (graded) commutator of coderivations. In fact, the Hochschild complex $C^\bullet$ is a differential graded Lie algebra with respect to this bracket.

In addition, the dot product (13) and the Hochschild differential (12) define the structure of a DG associative algebra on $C^\bullet(A,A)$.

3.2. The structure of a B_∞-algebra on the Hochschild complex

Define the structure of a B_∞-algebra on $C^\bullet(A,A)$ as follows:

$$\begin{aligned}
&M_0 := 0, && M_{0,1} = M_{1,0} := \mathrm{id},\\
&M_1 := d, && M_{0,n} = M_{n,0} := 0 \quad \text{for } n > 1,\\
&M_2(x_1,x_2) := x_1 \cdot x_2, && M_{1,n}(x; x_1, \dots, x_n) := \{x\}\{x_1, \dots, x_n\} \quad \text{for } n \ge 0,\\
&M_n := 0 \quad \text{for } n > 2, && M_{k,l} := 0 \quad \text{for } k > 1,
\end{aligned}$$

where $x, x_1, \dots, x_n \in C^\bullet(A,A)$.

Theorem 3.1 *These operations define the structure of a B_∞-algebra on the Hochschild complex $C^\bullet$.*

Remark 6 The braces were defined by Kadeishvili [14] and Getzler [9], the identities among them and the dot product were written down in [28], where this structure was called a homotopy G-algebra structure. The fact that this algebraic data defines a B_∞-structure was noticed by Getzler and Jones in [10].

Proof. Taking into account the vanishing operations M_n and $M_{k,l}$ and rewriting the rest in terms of the dot product and braces, the identities (6) through (8) can be simplified as follows.

The identities (6) for $n = 1$, 2, and 3, are equivalent to

$$d^2 = 0, \tag{14}$$

$$d(x_1x_2) = (dx_1)x_2 + (-1)^{|x_1|} x_1 dx_2, \tag{15}$$

$$(x_1x_2)x_3 = x_1(x_2x_3), \tag{16}$$

respectively.

The identities (7) are nontrivial only for $k = 1$, when they turn into the following.

$$\sum_{0 \le i_1 \le \cdots \le i_l \le m} (-1)^{\varepsilon} \{x\}\{z_1, \ldots, z_{i_1}, \{y_1\}\{z_{i_1+1}, \ldots\}, \ldots, z_{i_l}, \{y_l\}\{y_{i_l+1}, \ldots\}, \ldots, y_m\}$$
$$= \{\{x\}\{y_1, \ldots, y_l\}\}\{z_1, \ldots, z_m\}, \tag{17}$$

where $\varepsilon = \sum_{p=1}^{l} (|y_p| - 1) \sum_{q=1}^{i_p} (|z_q| - 1)$.

The identities (8) are nontrivial only when $k = 1$ and 2. For $k = 1$ they rewrite as the following family of identities:

$$\begin{aligned} d\{x\}\{y_1, \ldots, y_l\} &- (-1)^{|x|(|y_1|-1)} y_1 \cdot \{x\}\{y_2, \ldots, y_l\} \\ &+ (-1)^{|x|+|y_1|+\cdots+|y_{l-1}|+l-1} \{x\}\{y_1, \ldots, y_{l-1}\} \cdot y_l \\ = \{dx\}\{y_1, \ldots, y_l\} &- \sum_{i=0}^{l-1} (-1)^{|x|+|y_1|+\cdots+|y_i|-i} \{x\}\{y_1, \ldots, y_i, dy_{i+1}, \ldots, y_l\} \\ &- \sum_{i=0}^{l-2} (-1)^{|x|+|y_1|+\cdots+|y_{i+1}|-i} \{x\}\{y_1, \ldots, y_i, y_{i+1} \cdot y_{i+2}, \ldots, y_l\}, \end{aligned} \tag{18}$$

for each $l \ge 1$. For $k = 2$, Equations (8) turn into

$$\begin{aligned} \sum_{0 \le l_1 \le l} (-1)^{|x_2|(|y_1|+\cdots+|y_{l_1}|-l_1)} (\{x_1\}\{y_1, \ldots, y_{l_1}\}) \cdot (\{x_2\}\{y_{l_1+1}, \ldots, y_l\}) \\ = \{x_1 \cdot x_2\}\{y_1, \ldots, y_l\}, \end{aligned} \tag{19}$$

for each $l \ge 1$.

All these identities for the operations on the Hochschild complex may be checked directly. Some of the identities are classical, see e.g., Gerstenhaber [8], the others were not noticed until more recently, see [28]. One can find a detailed verification of the identities in Khalkhali's paper [16]. ■

Combining Theorems 2.1 and 3.1 and Corollary 2.2, we come to the following solution of Deligne's Conjecture.

Corollary 3.2 *The operad morphism* $\overline{E}^1 \to B_\infty$ *and the above action of* B_∞ *on the Hochschild complex* $C^\bullet$ *define on* $C^\bullet$ *the natural structure of an algebra over the operad* $\overline{E}^1$, *which is quasi-isomorphic to its homology* e_2.

Remark 7 This corollary along with Corollary 2.2 also yields the natural structure of a G_∞-algebra on $C^\bullet$, recovering a result of Tamarkin [26].

Remark 8 A complex of vector spaces with operations $x_1 \cdot x_2$ and $\{x\}\{x_1, \ldots, x_n\}$ for $n \ge 0$ satisfying identities (14)–(19) was called a *homotopy G-algebra* in [28, 12]. Kadeishvili rediscovered the same notion in [15] under the name of an *associative Hirsch algebra*.

4. Correction of [21]

As we have already mentioned, the paper [21] of Kimura, Zuckerman, and the author used the notion of a G_∞-algebra as an algebra over the Getzler-Jones' cellular operad, which was later noticed not to be cellular. The following changes have to be made to correct the resulting error.

In Section 4, after describing Getzler-Jones' cellular partition $K_\bullet \underline{\mathscr{M}}$, we should emphasize that it is not a cellular operad. However, there is a way to produce a DG operad out of it. Namely, notice that as a graded operad, $K_\bullet \underline{\mathscr{M}}$ is free on the collection of Fox-Neuwirth cells. The problem is that the differential is not well defined, in general. For those Fox-Neuwirth cells C_1 whose geometric boundary is the union of Getzler-Jones cells, the differential dC_1 is defined as the boundary operator. For those Fox-Neuwirth cells C_2 whose geometric boundary is not the union of Getzler-Jones cells, define the differential as a new generator dC_2. Take the free graded operad $K_\bullet$ generated by the Fox-Neuwirth cells and the differentials of the Fox-Neuwirth cells of the second type. The differential now is well defined on the generators: For a Fox-Neuwirth cell C_1 of the first type, the differential dC_1 is a linear combination of Getzler-Jones cells; for a Fox-Neuwirth cell C_2 of the second type, the differential is the generator dC_2; finally $d(dC_2) = 0$. This differential extends uniquely to the free graded operad $K_\bullet$. Definition 4.1 in [21] must be replaced by the following one.

Definition 4.1 *The* weak G_∞-operad *is the DG operad $K_\bullet$ constructed above. An algebra over it is called a* weak G_∞-algebra.

Every occurrence of the word "G_∞-algebra" in [21] must be replaced with the words "weak G_∞-algebra". Whenever the "operad" $K_\bullet \underline{\mathscr{M}}$ occurs in the sequel therein, it must be replaced with the above operad $K_\bullet$. All the lower identities written down in Sections 4.1 and 4.2 of [21] are satisfied in a weak G_∞-algebra and do not require corrections, because the Fox-Neuwirth cells used there are of the first type. For the same reason, the A_∞- and L_∞-operads map naturally to the weak G_∞-operad, therefore, a weak G_∞-algebra is naturally an A_∞- and L_∞-algebra. With the replacement of the G_∞-algebras by weak G_∞-algebras, all the results of the paper are correct with the same proofs. Moreover, Conjecture 2.3 of [21] with the above change has been proven in the meantime by Yi-Zhi Huang and Wenhua Zhao [13]. Thus it must be renamed to a theorem, as follows.

Theorem 4.1. (Huang and Zhao) *Let $V^\bullet[\cdot]$ be a TVOA satisfying $G(0)^2 = 0$. Then the dot product and the skew-symmetrization of the bracket defined in [21] can be extended to the structure of a weak G_∞-algebra on a certain completion of $V^\bullet[\cdot]$.*

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Linearization problems for Lie algebroids and Lie groupoids

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Abstract. In this survey, we discuss a series of linearization problems — for Poisson structures, Lie algebroids, and Lie groupoids. The last problem involves a conjecture on the structure of proper groupoids. Attempting to prove this by the method of averaging leads to problems concerning almost actions of compact groups and almost invariant submanifolds for compact group actions. The paper ends with a discussion of possible extensions of the convexity theorems for momentum maps of hamiltonian actions of compact groups.

Keywords: Poisson structure, Lie groupoid, Lie algebroid, linearization, group action, invariant submanifold, averaging

Mathematics Subject Classifications (2000): 22A22, 53D17, 58D19

In memory of Moshé Flato, 1937-1998

1. Introduction

Why is it so hard to prove the linearizability of Poisson structures with semisimple linear part? Conn published proofs about 15 years ago in a pair of papers [5, 6] full of elaborate estimates. Except for the somewhat more conceptual reformulation by Desolneux-Moulis [9] in the smooth case, no simplification of Conn's proofs has appeared.

This is a mystery to me, because analogous theorems about the linearizability of actions of semisimple groups near their fixed points were proven in the compact (smooth or analytic) case by Bochner [3] using a simple averaging method and in the noncompact analytic case by Guillemin-Sternberg [13] and Kushnirenko [18], who used analytic continuation from the compact case — a nonlinear version of "Weyl's unitary trick". Hermann [16] established formal linearization for actions of general semisimple algebras, using cohomological methods similar to those which will appear several times in the present report.

After Conn's work appeared, I tried without success to prove his results by simple averaging. In a conversation in 1994 over coffee in the Jardin du Luxembourg, Moshé Flato and Daniel Sternheimer revived my interest in the subject. Their experience with Jacques Simon on other linearization problems (see for instance [10]) led them to expect simpler proofs of Conn's theorems. In addition, they pointed out that Poisson linearization could be seen as an infinite-dimensional

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Levi decomposition. Although I expressed strong skepticism at the time, Moshé and Daniel's optimism has remained in my mind and kept me from abandoning the problem. In particular, it has resulted in new investigations which, although they have not produced a Poisson linearization theorem (and maybe never will!), have led to a few results and many questions which are interesting in their own right.

These new results and problems form the content of the present paper, which I offer as a small memorial to Moshé. The paper also serves to record a talk which I gave at the Dijon Mathematical Physics Seminar in June, 1999, at Daniel's invitation.[1] The occasion was much saddened by Moshé's absence, but it has been gratifying to see the Seminar continue as an ongoing international "workshop" for discussion of many of Moshé's favorite scientific issues.

2. Linearization and Levi decomposition

The linearization of a Poisson structure with semisimple linear part can be seen, as mentioned above, as something like a Levi decomposition of a Lie algebra of functions with the Poisson bracket operation. Here are the details.

Let $\pi = \frac{1}{2}\sum(\sum c_{ij}^k x_k + \mathbf{O}(x^2))\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$ be a Poisson structure defined on a neighborhood $\mathscr{U}$ of the origin in $\mathbb{R}^n$, i.e., $\{x_i, x_j\} = \sum c_{ij}^k x_k + \mathbf{O}(x^2)$. In the Lie algebra of germs at 0 of smooth functions, the germs of functions vanishing at 0 form a Lie subalgebra $\mathfrak{m}$ in which those vanishing to order at least 2 form a Lie algebra ideal $\mathfrak{m}^2$. The quotient $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{g}$ may be identified with the cotangent space of $\mathbb{R}^n$ at 0, the (finite-dimensional) **cotangent Lie algebra** whose structure constants are just the first Taylor coefficients c_{ij}^k of the Poisson structure π. The **linearization problem** is to find (perhaps after shrinking the neighborhood $\mathscr{U}$) new coordinates $(y_1, \ldots, y_n)$ centered at 0 such that $\{y_i, y_j\} = \sum c_{ij}^k y_k$ without any higher order terms.

To view a solution of the linearization problem for semisimple $\mathfrak{g}$ as a Levi decomposition, we note first that, in the exact sequence of Lie algebras

$$0 \to \mathfrak{m}^2 \to \mathfrak{m} \to \mathfrak{g} \to 0,$$

the quotient $\mathfrak{g}$ is (by assumption) semisimple, while the kernel $\mathfrak{m}^2$ has a certain nilpotency property. By this last statement, we mean that the spaces $\mathfrak{m}^k$ of functions vanishing to order at least k have the property that $[\mathfrak{m}^2, \mathfrak{m}^k] \subseteq \mathfrak{m}^{k+1}$. Unfortunately, the intersection $\mathfrak{m}^\infty = \bigcap_k \mathfrak{m}^k$ is not zero, but consists of the germs of functions which vanish to infinite order at the origin.[2]

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[1] An earlier version of this paper was presented in a lecture at the Chern Symposium at MSRI [24].

[2] There are some interesting questions to be answered about the Lie algebra $\mathfrak{m}^\infty$. Does it have any semisimple quotients? Does it have any finite-dimensional quotients at all? Equivalently, does it have any finite-dimensional representations? What are the its finite-dimensional subalgebras?

A "Levi decomposition" of $\mathfrak{m}$ should be a vector space direct sum decomposition $\mathfrak{m} = \mathfrak{h} \oplus \mathfrak{m}^2$ in which $\mathfrak{h}$ is a subalgebra or, equivalently, a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{m}$ which splits the exact sequence.

Given a linearization $(y_1, \ldots, y_n)$ of the Poisson structure, the subalgebra spanned by the germs of the y_i is the required $\mathfrak{h}$. Conversely, given a Levi decomposition and letting $(\eta_1, \ldots, \eta_n)$ be the preimages in $\mathfrak{h}$ of the basis $(x_1 + \mathfrak{m}^2, \ldots, x_n + \mathfrak{m}^2)$ of $\mathfrak{g}$, representatives y_i of the germs η_i have a common domain on which they solve the linearization problem.

Note that everything above remains true if the algebra of smooth germs is replaced by either that of real analytic germs or that of formal power series, with the additional feature that $\mathfrak{m}^\infty$ is now reduced to zero.

Unfortunately, I do not know of any proof of the standard Levi decomposition theorem which would apply here, so it seems that we have only reformulated the linearization problem. Nevertheless, this reformulation leads to some interesting new questions.

Finally, we note that A. Wade [22] has found a formal normalization of Poisson structures at general singular points which is based on the Levi decomposition of the cotangent Lie algebra itself.

3. Lie algebroids

The linearization problem for Poisson manifolds has analogues for Lie algebroids and Lie groupoids. To describe them, we recall that a Lie algebroid over a manifold M is a vector bundle $A \to M$ with a Lie algebra structure (over $\mathbb{R}$) on its space $\Gamma(A)$ of smooth sections and a bundle map $\rho : A \to TM$ inducing a Lie algebra homomorphism from sections of A to vector fields on M, such that $[a, fb] = f[a,b] + (\rho(a) \cdot f)b$ for sections a and b and functions f. Locally (which is all we need for this paper), a Lie algebroid over an open subset $\mathscr{U}$ of $\mathbb{R}^n$ is specified by the bracket relations $[e_i, e_j] = \sum c_{ij}^k(x) e_k$ among a basis of sections of A and the component description $\rho(e_i) = \sum b_{ij}(x) \frac{\partial}{\partial x_j}$ of the anchor map. The axioms of a Lie algebroid then become differential equations relating the **structure functions** c_{ij}^k and b_{ij}.

As with Poisson manifolds, the local classification of Lie algebroids can be reduced by a splitting theorem (modeled on the splitting theorem of Dazord [7] for singular foliations) to the totally singular case where the anchor map ρ vanishes at a point. In coordinates centered at such a point, we may write $c_{ij}^k(x) = c_{ij}^k(0) + \mathbf{O}(x)$ and $b_{ij}(x) = \frac{\partial b_{ij}}{\partial x_k}(0) x_k + \mathbf{O}(x^2)$. Our problem is to present the Lie algebroid in such a way that the higher order terms in the structure functions disappear. Although we have more conditions to satisfy than in the Poisson linearization problem, we also have more variables at our disposal, since we can change the basis of sections as well as the coordinates on the base.

We first try to change the basis to reduce the functions $c_{ij}^k(x)$ to the constants $c_{ij}^k(0)$. This is equivalent to making the Lie algebroid into one of a special kind. Namely, the constants $c_{ij}^k(0)$ define a Lie algebra structure on the fibre of A at 0, and the vector fields $\rho(e_i)$ give an action of this Lie algebra on $\mathscr{U}$. Conversely, when a Lie algebra $\mathfrak{g}$ acts on a manifold M, there is a Lie algebroid structure on the trivial bundle $\mathfrak{g} \times M$ for which the anchor is given by the action and the constant sections form a subalgebra on which the bracket is that of $\mathfrak{g}$. So our task is to determine when a given Lie algebroid is, near a point where the anchor vanishes, such an **action Lie algebroid**. If this task is accomplished, we can finish simplifying the Lie algebroid structure functions by linearizing the action, under the usual hypotheses of semisimplicity and, in the smooth setting, compact type.

Making A into an action Lie algebroid on a neighborhood amounts precisely to finding a Lie algebra homomorphism which splits the exact sequence:

$$0 \to \text{sections vanishing at } 0 \to \text{sections of } A \to \text{fibre at } 0 \to 0,$$

which is analogous to the sequence in the Poisson linearization problem. In particular, if we pass to germs, the kernel of the restriction map is once again "topologically nilpotent" in the sense discussed in Section 2. The analogy is so close that it is tempting to transfer the proofs of Poisson linearization theorems to the Lie algebroid case. In the formal category, this is no problem. The obstructions to stepwise lifting from the fibre $\mathfrak{g}$ at 0 to higher and higher jets of sections lie in the spaces $H^2(\mathfrak{g}, S^k(\mathbb{R}^n) \otimes \mathfrak{g})$, where $\mathfrak{g}$ acts on itself by the adjoint representation and on the symmetric tensor power $S^k(\mathbb{R}^n)$ via the action on $\mathbb{R}^n$ given by $e_i \mapsto \frac{\partial b_{ij}}{\partial x_k}(0)$. These cohomology spaces vanish when $\mathfrak{g}$ is semisimple.[3]

It seems likely that Conn's methods can be extended to establish linearizability in the smooth and analytic settings. I have not yet tried to do this, however, preferring another approach using averaging which might eventually lead to a new proof of Poisson linearization. The following sections describe this approach.

4. Lie groupoids

Averaging requires an action of a compact Lie group, not just of its Lie algebra. We therefore pose a linearization problem for Lie groupoids, which are the "integrated" form of Lie algebroids (see [4] or [19]). Note, though, that not every Lie algebroid is integrable to a groupoid [1], so that linearization of Lie groupoids will linearize only certain Lie algebroids.

Let Γ be a Lie groupoid over M with source and target maps β and α; the product gh is defined when $\beta(g) = \alpha(h)$. Γ is a **proper groupoid** if $(\alpha, \beta) : \Gamma \to M \times M$ is a proper mapping and if the source and target maps are locally trivial fibrations. This term, like many other terms in groupoid theory, comes from the following example.

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[3] For formal Poisson linearization, the relevant cohomology spaces are $H^2(\mathfrak{g}, S^k(\mathfrak{g}))$.

If G is a group acting on M, then $G \times M$ is a Lie groupoid over M with $\alpha(g,x) = gx$, $\beta(g,x) = x$, and product $(g,hx)(h,x) = (gh,x)$. We call this the **action groupoid** associated to the group action. It is a proper groupoid if and only if the action is a proper action.

For a general groupoid Γ over M, the **isotropy group** Γ_x of x in M is defined as $\alpha^{-1}(x) \cap \beta^{-1}(x)$. Since $\Gamma_x = (\alpha,\beta)^{-1}(x,x)$, every isotropy group in a proper groupoid is compact.

A subset $\mathscr{U} \subseteq M$ is called **invariant** under Γ if $\alpha(\beta^{-1}(u)) \subseteq \mathscr{U}$. An invariant point is called a **fixed point**.

We show in [26] that every neighborhood of a fixed point of a proper groupoid contains an invariant neighborhood, so that the study of proper groupoids can be localized around fixed points. We make the following conjecture.

Proper Groupoid Structure Conjecture. *If x is a fixed point of a proper groupoid Γ, then x has an invariant neighborhood on which Γ is isomorphic to the action groupoid associated to some action of the compact group Γ_x.*

If the "PGS" conjecture above is true, the next step is to linearize the Γ_x action, which would give an equivalence between proper groupoids near their fixed points and linear actions of compact groups. From here it should be possible, using a slice theorem, to get a normal form for a proper groupoid in the neighborhood of any orbit $\beta(\alpha^{-1}(x))$. This program is described in [26].

The PGS conjecture fits into the Levi decomposition picture presented in Section 2 via the exact sequence of groups:

$$1 \to \mathscr{G}^1_{\mathscr{U}} \to \mathscr{G}_{\mathscr{U}} \to \Gamma_x \to 1.$$

Here, $\mathscr{G}_{\mathscr{U}}$ is the group of **admissible sections** of Γ over the invariant set $\mathscr{U}$, i.e., the smooth submanifolds σ of the restricted groupoid $\Gamma_{\mathscr{U}}$ for which the restrictions $\alpha|_\sigma$ and $\beta|_\sigma$ are diffeomorphisms from σ to $\mathscr{U}$. The map $\mathscr{G}_{\mathscr{U}} \to \Gamma_x$ is evaluation at x, and the kernel $\mathscr{G}^1_{\mathscr{U}}$ consists of those sections which meet the unit section at x. Finding an isomorphism of $\Gamma_{\mathscr{U}}$ with an action groupoid amounts to finding a cross section $\Gamma_x \to \mathscr{G}_{\mathscr{U}}$ which is a group homomorphism.

Since the properness of Γ implies that Γ_x is compact, we could construct a homomorphic section by averaging if the kernel $\mathscr{G}^1_{\mathscr{U}}$ were the additive group of a vector space. Of course, this is not the case, but we do get a nice composition series for $\mathscr{G}^1_{\mathscr{U}}$ if we pass to infinite jets at x of admissible sections of Γ. We can then use the standard stepwise proof (this time using group cohomology) to find an action groupoid structure for Γ over a "formal neighborhood of x". Note that semisimplicity plays no role here — compactness of Γ_x is enough, so it could be a torus, for instance.

In the analytic or smooth categories, we could try to imitate Conn's proofs, but instead we propose another approach. We confine our discussion to the smooth case.

Since α is a locally trivial fibration, we can choose a cross section $\sigma_0 : \Gamma_x \to \mathscr{G}_{\mathscr{U}}$ for sufficiently small $\mathscr{U}$ such that the corresponding map from $\Gamma_x \times \mathscr{U}$ to $\alpha^{-1}(\mathscr{U})$ is a diffeomorphism. This σ_0 will not generally be a group homomorphism, so we will try to "improve" it to become one. By the formal normal form result, we can assume (here we imitate Conn [6]) that σ_0 is a homomorphism modulo flat sections; i.e., for each g and h in Γ_x, $\sigma_0(g)\sigma_0(h)$ and $\sigma_0(gh)$ are tangent to infinite order at the point gh where they meet Γ_x. Using our local trivialization of α and local coordinates on a sufficiently small open neighborhood $\mathscr{U}$ of x, we can assume that, for all g and h in Γ_x, $\sigma_0(g)\sigma_0(h)$ and $\sigma_0(gh)$ considered as maps from $\mathscr{U}$ to Γ_x are as close as we wish, together with as many derivatives as we desire.

Thus, giving $\mathscr{G}_{\mathscr{U}}$ a C^k-metric, we can say that σ_0 is an "almost homomorphism" in the sense that $d(\sigma_0(g)\sigma_0(h), \sigma_0(gh)$ is small for all g and h in Γ_x, and our problem is to approximate this almost homomorphism by a homomorphism.

5. Almost homomorphisms and almost invariant submanifolds

There are quite a few theorems which assert that an almost homomorphism is near a homomorphism (a very general formulation of this problem was proposed by Ulam [21]), but none of them suits our needs, so we must prove new ones.

The following two theorems are closest to what we are looking for.

Almost Homomorphism Theorem. (Grove-Karcher-Ruh [12]). *Let G and H be compact Lie groups. Choose a bi-invariant metric on H such that: (1) the exponential map is an embedding when restricted to the open ball of radius π in the Lie algebra $\mathfrak{h}$; (2) the Lie bracket on $\mathfrak{h}$ satisfies $||[v,w]|| \leq ||v||\ ||w||$. (Such a metric always exists.) Let $\sigma_0 : G \to H$ be a continuous map such that $d(\sigma_0(g)\sigma_0(h), \sigma_0(gh)) \leq q \leq \pi/6$ for all g and h in G. Then there is a (continuous) homomorphism $\sigma : G \to H$ such that $d(\sigma_0(g), \sigma(g)) < 1.36q$ for all g in G.*

Almost Representation Theorem. (de la Harpe-Karoubi [8]) *Let T_0 be a continuous map from a compact group G to the group H of invertible bounded linear operators on some Hilbert space. Let $K \geq 1$ and $\varepsilon \leq 2^{-6}$ be real numbers such that $||T_0(gh) - T_0(g)T_0(h)|| \leq \varepsilon(2K)^{-9}$ for all g and h in G. Then there is a continuous homomorphism $T : G \to H$ such that $||T_0(g) - T(g)|| \leq \varepsilon$ for all g in G.*

An intermediate result between those theorems and the one we want for groupoid linearization would be the following, stated somewhat imprecisely.

Almost Action Conjecture. *Let G be a compact group and H the group of diffeomorphisms of a compact manifold M. If $\sigma : G \to H$ is a map such that the distance $d(\sigma_0(g)\sigma_0(h), \sigma_0(gh)$ is sufficiently small for all g and h in G, then there is a homomorphism ψ close to σ. Here, the distance between two diffeomorphisms is taken to be their C^1 distance defined using a riemannian metric μ, and the measure of "sufficient smallness" will depend on the size of the first and second derivatives* *of the maps in* $\phi(G)$*, as well as on upper bounds on the absolute value of the curvature and the reciprocal of the injectivity radius for the metric* μ.

Even this conjecture is currently beyond our reach. When G is finite, it should follow from the following result in [25].

Almost Invariant Submanifold Theorem. *Let* N *be a compact submanifold of a riemannian manifold* M*, and let* G *be a compact group acting on* M*. Choose an invariant metric on* M *such that: (1) the exponential map of* M*, restricted to the normal bundle of* N*, is (defined and) an embedding on the open ball bundle* B *of radius 1; (2) all the sectional curvatures of* M *on* $\exp(B)$ *have absolute value less than 1; (3) the exponential map of* M *is (defined and) an embedding on the ball of radius 1 in each tangent space of* $\exp(B)$*. (This is always possible: Multiply any invariant metric by a suitably large constant.) If the* C^1 *distance (defined below)* $d(N,gN)$ *is less than* $\varepsilon < \frac{1}{20000}$ *for each* $g \in G$*, then there is a* G*-invariant submanifold* $\overline{N} \subseteq M$ *such that* $d(N,\overline{N}) < 136\sqrt{\varepsilon}$.

The distance $d(N,N')$ between submanifolds in the theorem above is defined as follows. First of all, we assume that N' is the image under exp of a section s of the normal bundle of N. For each $x \in N$, we take the maximum $a(x)$ of the following two numbers: The length of the geodesic segment σ from x to $s(x)$, and the maximum angle between unit vectors in T_xN and $T_{x'}N'$, where vectors in one space are moved to the other by parallel transport along σ. Now $d(N,N')$ is defined as $\max_{x\in N} a(x)$.

Like the proofs of the Almost Homomorphism Theorem and the Almost Representation Theorem, the proof of the Almost Invariant Submanifold Theorem uses averaging over the group G. The estimates involved are nontrivial, but they are somewhat simpler, and certainly more geometric, than the ones in Conn's proof. The challenge now is to work back from here to prove some linearization theorems without adding too much more complication.

6. Properness and convexity

Our interest in proper groupoids was originally motivated, not by linearization problems, but by an attempt to understand the convexity theorems of Atiyah [2], Guillemin-Sternberg [14, 15] and Kirwan [17] as results in Poisson geometry. Their theorems establish convexity properties of the image of the momentum map $J : S \to \mathfrak{g}^*$ for a hamiltonian action of a compact Lie group G on a symplectic manifold S. When G is a torus, the theorem states that $J(S)$ is a convex polyhedron, while for general compact G it is the intersection of $J(S)$ with a positive Weyl chamber which is convex.

When G is simply connected, a momentum map is simply a Poisson map from S to $\mathfrak{g}^*$, which suggests that there might be a convexity theorem for Poisson maps $J : S \to P$ from symplectic manifolds to a wider class of Poisson manifolds. The case of the torus $\mathbb{T}^n$ shows that the problem involves more than just Poisson manifolds: In this case, $P = \mathscr{T}^{n*}$ has the zero Poisson structure, so that the set of Poisson maps from S to P is closed under arbitrary diffeomorphisms of P, which usually destroy any convexity properties of the image. The point here is that J must the momentum map for an action of $\mathbb{T}^n$ and not just of its universal covering $\mathbb{R}^n$.

To express in geometric terms the choice of group associated to a given (dual of a) Lie algebra, we recall that hamiltonian actions of the Lie group G correspond to symplectic actions of the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ [20]. (The essentials of this result were presented in [23], before the advent of symplectic groupoids in the 1980's.) We therefore pose the following conjecture (having well in mind that some further hypotheses may be needed.)

Poisson Convexity Conjecture. *Let $\Gamma \rightrightarrows P$ be a proper symplectic groupoid over the Poisson manifold P, and let $J : S \to P$ be the moment map (see [20]) for an action of Γ on a symplectic manifold S. Then the image $J(S) \subseteq P$ has a convexity property with respect to some affine structure attached to the groupoid Γ.*

The problem here is to establish the appropriate notion of convexity, which requires an understanding of the structure of proper symplectic groupoids. Such an understanding would also tell us the extent to which our conjecture goes beyond the known convexity theorems for group actions.

Examples. Let G be a semisimple Lie group of noncompact type. The **elliptic** subject $E \subset \mathfrak{g}^*$ is defined to consist of those elements whose coadjoint isotropy group in G is compact. (Under the orbit method, E corresponds to the discrete series of representations of G.) It appears that E is an open subset on which the restriction of the coadjoint representation is a proper action. By restricting the symplectic groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ to E, we obtain a proper symplectic groupoid for E. Now a hamiltonian action of G on S may be called "elliptic" if the image $J(S)$ of the momentum map lies in E. A special case of our problem would be to establish a convexity theorem for the momentum maps of such elliptic actions.

The previous example is still a transformation groupoid. Another example of a proper symplectic groupoid is the fundamental groupoid $\pi(M)$ of a compact, connected, symplectic manifold M with finite fundamental group. Symplectic actions of this groupoid correspond to symplectic actions of the fundamental group. (See Section 7.6 of [4].) In this case, there is no convexity problem, since momentum maps to the symplectic manifold M are surjective, but perhaps products $\pi(M) \times T^*G \rightrightarrows M \times \mathfrak{g}^*$ will give interesting examples.

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Appendix A: Program of the scientific conference

Quantization, deformations, and symmetries

Université de Bourgogne, Dijon, September 6 – 8 1999

Monday September 6

9:15 – 9:30	Opening of the special session and of the scientific conference by Arthur Jaffe and Daniel Sternheimer
9:30 – 14:00	Get together session with informal discussions and organization of seminars and poster sessions; lunch
14:30 – 15:25	Yakir Aharonov (U. South Carolina and Tel-Aviv) *A new approach to quantum mechanics* Chair: Paul Sally (Chicago)
15:40 – 16:35	Elliott Lieb (Princeton) *Stability of matter and the self-energy of an electron in non-perturbative quantum electrodynamics* Chair: Marcel Guenin (Geneva)
	– Coffee break –
17:15 – 18:10	Sergio Ferrara (CERN and UCLA) *Superconformal field theories, multiplet shortening, and the* $AdS_5/SCFT_4$ *correspondence* Chair: Robert Finkelstein (UCLA)
18:25 – 19:20	Christian Frønsdal (UCLA) *Singletons and neutrinos* Chair: Piotr Rączka (Warsaw)

Tuesday September 7

9:15 – 10:10	Leon Takhtajan (SUNYSB) *Quantum field theories on an algebraic curve* Chair: J.-P. Labesse (Director of CIRM)
	– Coffee break –
10:35 – 11:30	Maxim Kontsevich (IHES) *Geometry of the moduli space of non-commutative algebras* Chair: Hideki Omori (Science University of Tokyo)
11:45 – 12:40	Alain Connes (Collège de France) *Cyclic cohomology and Hopf algebras* Chair: Murray Gerstenhaber (U. Pennsylvania)

13:00 – 15:00	*Lunch and seminars by T. Kimura (Boston), M. Bordemann and S. Waldmann (Freiburg)*
15:00 – 15:55	Simone Gutt (ULB and Metz) *Variations on deformation quantization* Chair: Alan Weinstein (Berkeley)
16:10 – 17:05	Boris Fedosov (Moscow and Potsdam) *On G-trace and G-index in deformation quantization* Chair: Ryszard Nest (Copenhagen)
	– Coffee break –
17:45 – 18:40	Louis Boutet de Monvel (Paris 6) *Real and complex star products* Chair: Yvette Kosmann-Schwarzbach (École polytechnique)
18:50 – 20:00	*Seminars by A. Karabegov (Erevan), A. Asada (Shinshu), and R. Léandre (Nancy); poster sessions*

Wednesday September 8

9:15 – 10:10	Jacques C. H. Simon (U. Bourgogne) *Nonlinear relativistic evolution equations:* *Survey of a new approach* Chair: Ivan Todorov (Sofia)
	– Coffee break –
10:35 – 11:30	Bertram Kostant (MIT) *A cubic Dirac operator and a generalization of the Bott-Borel-Weil theorem* Chair: Nicolai Reshetikhin (Berkeley)
11:45 – 12:40	Wilfried Schmid (Harvard) *L-functions attached to Maass modular forms* Chair: Gérard Schiffmann (Strasbourg)
13:00 – 15:00	*Lunch and seminars by P. Rączka (Warsaw), J. Niederle (Prague), and M. Kontsevich (IHES)*
15:00 – 15:55	Jürg Fröhlich (ETHZ) *Classical limits of Bose systems* Chair: Jiri Niederle (Prague)
16:10 – 17:05	Ludwig Faddeev (St. Petersburg and Helsinki) *The modular double of a quantum group* Chair: Yuri Novozhilov (St. Petersburg)
17:30 – 19:00	Closure by Gilles Bertrand, vice president for research and former president of the University, and cocktail

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